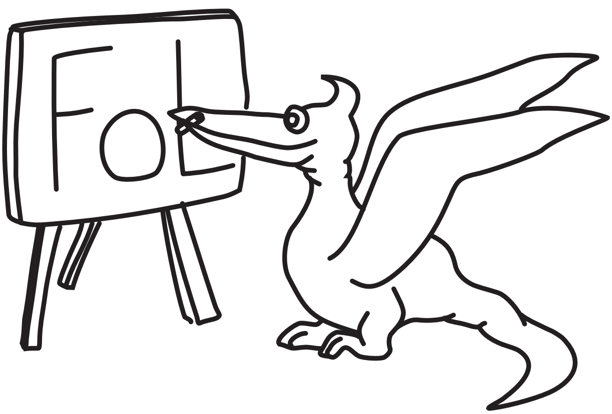
Solution of 3rd Online Physics Brawl



Problem FoL.1 ... jumping dog

An escaping prisoner needed to jump from one rooftop to another one because it's just the thing escaping prisoners do. The first building, from which he jumped, is $H = 16\,\mathrm{m}$ *tall and the second one is* $h = 11.6\,\mathrm{m}$*, the buildings are* $d = 4\,\mathrm{m}$ *apart. The velocity of the prisoner at the moment of jump is* $v = 3.8\,\mathrm{m\cdot s^{-1}}$ *and he is jumping parallel to the sufrace. Determine the missing/redundant distance (using the -/+ signs) after the jump relative to the rooftop of the second building. Air resistance is negligible.Assume gravitational acceleration* $g = 9.81\,\mathrm{m\cdot s^{-2}}$*.*

Kiki has found and remade this HRW problem while drinking tea.

First, we estimate the time needed for the jump, $t = \sqrt{2\Delta h/g}$ where Δh is the height difference between the two buildings. Knowing the time, we can compute the distance jumped on the x-axis $x = vt\cos\alpha$, where $\alpha = O$.

$$x = vt\cos\alpha$$

$$x = v\sqrt{\frac{2\Delta h}{g}}$$

Filling in the numbers, we get $x \doteq 3.60\,\mathrm{m}$, meaning that the prisoner would need $0.40\,\mathrm{m}$ more to reach the second rooftop, hence we write the result as $\Delta \doteq -0.40\,\mathrm{m}$.

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Problem FoL.2 ... source of radiation

Compute the wavelength of radiation passing through an optical grid. The distance of the optical grid from the projection screen is $2.0\,\mathrm{m}$*, the distance between* 0^{th} *and* 2^{nd} *maximum is* $6.0\,\mathrm{cm}$*. Period of the optical grid is* $5.0\cdot 10^{-6}\,\mathrm{m}$*. The result should be in nanometers.*

Monika couldn't believe her eyes.

Diffraction on the optical grid is described by the following relation

$$\sin\alpha = \frac{k\lambda}{a}\,,$$

where α is an incidence angle of the ray measured from the perpendicular, k is an order of the maximum (here $k = 2$), λ is the wavelength of the radiation and a is the period of the grid. The distance from the projection screen to the optical grid is l and the distance between 0^{th} and 2^{nd} maximum is b. For $b \ll l$, we can write $\sin\alpha \approx b/l$. Knowing this, we can use the relation for the diffraction and calculate the wavelength:

$$\lambda \approx \frac{ab}{kl} = 75\,\mathrm{nm}\,.$$

The radioation has $\lambda \doteq 75\,\mathrm{nm}$, hence it falls into the ultraviolet part of electromagnetic spectrum.

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Problem FoL.3 ... fluctuational

Quantum electrodynamics which connects quantum theory with the relativity brought revolution into our understanding of the vacuum. Vacuum isn't emptiness and nothingness anymore, it's a state with the lowest energy. The lowest possible energy state cannot be zero since we have to take into account the uncertainty principle – which leads us to the fluctuations of the vacuum. The strongest mechanical demonstration of quantum fluctuations is the attraction force between two mirrors (two parallel metallic plates without any charge) separated by a narrow gap. Around the system, there are waves with all frequencies but inside the gap, there are only such waves which correspond to the resonance frequency of the gap. This results in a small, yet measurable force which pushes the mirrors closer to each other. It's magnitude depends on the area of the mirrors S (it's natural to assume linear dependence here) and the width of the gap d. There are two fundamental constants in the relation, c and h:

$$F = Kc^{\alpha}h^{\beta}d^{\gamma}S\,.$$

Using dimensional analysis, determine the coefficients α, β and γ and the force for the values $S = 1\,\mathrm{cm}^2$ and $d = 1\,\mu\mathrm{m}$. K is a dimensionless constant, its value is $K = \pi/480$ as can be derived with precise calculations.

Zdeněk went through his old excercise book and was surprised to no end.

Dimensional analysis results in following equation

$$\mathrm{kg}^1\mathrm{m}^1\mathrm{s}^{-2} = \mathrm{m}^{\alpha+2\beta+\gamma+2}\mathrm{s}^{-\alpha-\beta}\mathrm{kg}^{\beta}\,.$$

Comparing the coefficients results in $\alpha = 1$, $\beta = 1$ and $\gamma = -4$. Substituing these numbers into the given relation for F, we obtain

$$F = \frac{\pi hcS}{480d^4}\,.$$

With the given values, the force is $F \doteq 1.3 \cdot 10^{-7}\,\mathrm{N}$. This effect is known as Casimir effect.

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Problem FoL.4 ... czech energetics

Imagine you transform a 500 Euro banknote into energy. How many times more valuable does it become? The banknote weights 1.1 g, 1 kWh of electric energy costs 20 cents. Round the results to tens.

Try to discover the real value of money.

The energy gained from the banknote van be computed using the relation $E = mc^2$. For the 500 euro banknote, it's 27.5 GWh. Multiplying by the price for one kWh and dividing by the nominal value of the banknote, the banknote becomes 11 000 times more valuable than it's nominal value.

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Problem FoL.5 ... fall one

What is the stabilized velocity of a falling leaf, if the leaf has enough time to reach an equilibrium. Consider the grammage of the leaf similar to be to that of office paper ($80\,\mathrm{g\cdot m^{-2}}$) and that it has the shape is of a hemisphere. Consider Newton's relation for air resistance and take air density to be $\varrho = 1.29\,\mathrm{kg\cdot m^{-3}}$ and the drag coefficient $C = 0.33$. Assume gratitational acceleration $g = 9.81\,\mathrm{m\cdot s^{-2}}$. Those who are afraid of falling leaves have a guilty conscience.

Putting the air resistance force and gravity to equality, $C\varrho Sv^2/2 = mg$ and considering that from the knowledge of grammage, we can compute the mass $m = \sigma S$, we express the stabilized velocity as

$$v = \sqrt{\frac{2\sigma g}{C\varrho}} = 1.9\,\mathrm{m\cdot s^{-1}}\,.$$

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Problem FoL.6 ... choo choo train

A little train named Denis decided to go through one turn repetitively, speeding up till he derails. The radius of the turn is $R = 190\,\mathrm{m}$, tilt of the rail is $\alpha = 5^\circ$, the rail width is $d = 1.4\,\mathrm{m}$ and the center of the mass of the train is $h = 1.6\,\mathrm{m}$ from the rails. What's the difference between minimal and maximal speed with which the train could go through the turn without derailing. The result should be in kilometres per hour. The gravitational acceleration is $g = 9.81\,\mathrm{m\cdot s^{-2}}$. The train moves horizontally. Jakub wants a driving licence for a train.

First of all, let's find out the minimum velocity allowed, using the angle between the perpendicular of the train and the abscissa between the center of the mass of the train and the rail. From the geometrical point of view, φ is

$$\varphi = \operatorname{arctg}\frac{d}{2h} \doteq 23^\circ\,,$$

which is more than the slope of the rail, $\alpha = 5^\circ$. Hence the train can stand on the rail and won't fall off. So the minimum velocity is zero. The maximal velocity $v_{\max}$ will occur in the moment when the total force acting on the train will point from the center of the mass of the train towards the outer rail. The angle between total acting force and the gravitational force is $\alpha + \varphi$. Hence for the ratio of centrifugal and gravitational force, we can write

$$\frac{F_\mathrm{o}}{G} = \frac{\frac{mv^2}{R}}{mg} = \operatorname{tg}(\alpha + \varphi)\,,$$

from which we can derive the maximal velocity

$$v_{\max} = \sqrt{Rg\operatorname{tg}(\alpha + \operatorname{arctg}\frac{d}{2h})} \doteq 115\,\mathrm{km\cdot h^{-1}}\,,$$

which is also the difference between the minimal and maximal velocities, since the minimal one is zero.

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Problem FoL.7 ... collision!!!

A positron and a helium nucleus are approaching each other, both are moving in a straight line, with the same speed $v = 2\,000\,\mathrm{km\cdot s^{-1}}$. What will the distance be between them in the moment when the positron stops (in the reference frame of the lab)? Express in pm. *Solve it clasically.*

Tomáš Bárta wanted to assign a particle physics task.

At the beginning, the mass of the positron is m_e and its velocity is $v_{\mathrm{e},0} = v$, for the helium nucleus, we have mass m_H and velocity $v_{\mathrm{H},0} = -v$ (it is aimed in the opposite direction). The moment we are interested in is when the velocity of positron decreases to zero and when it occurs the velocity of the positron is $v_{\mathrm{e},1} = 0$ and the velocity of the helium nucleus $v_{\mathrm{H},1}$. From the conservation of momentum, we can write

$$m_\mathrm{e} v_{\mathrm{e},0} + m_\mathrm{H} v_{\mathrm{H},0} = m_\mathrm{e} v_{\mathrm{e},1} + m_\mathrm{H} v_{\mathrm{H},1}\,.$$

Plugging in the numbers, we arrive at

$$v_{\mathrm{H},1} = v\frac{m_\mathrm{e} - m_\mathrm{H}}{m_\mathrm{H}}\,.$$

Then, using the conservation of energy, we can derive the distance of the particles. Assume zero potential energy at the beginning, since the distance between the particles was infinite. Hence

$$\frac{1}{2} m_\mathrm{e} v_{\mathrm{e},\mathrm{O}}^2 + \frac{1}{2} m_\mathrm{H} v_{\mathrm{H},\mathrm{O}}^2 = \frac{1}{2} m_\mathrm{e} v_{\mathrm{e},1}^2 + \frac{1}{2} m_\mathrm{H} v_{\mathrm{H},1}^2 + k\,\frac{Q_1 Q_2 e^2}{d}\,,$$

where k is Coulomb constant, Q_1 and Q_2 are the charges of positron and of helium nucleus respectively in the multiples of an elementary charge (($Q_1 = 1$, $Q_2 = 2$), e is the elementary charge and d is the distance between the particles. Plugging in the known velocities,

$$\frac{1}{2}(m_\mathrm{e} + m_\mathrm{H})v^2 = \frac{1}{2}v^2\frac{(m_\mathrm{H} - m_\mathrm{e})^2}{m_\mathrm{H}} + k\,\frac{Q_1 Q_2 e^2}{d}\,.$$

In the case of these equations, we know all the variables' values, so we can plug in the numbers and

$$d = \frac{2k\,Q_1 Q_2 e^2}{v^2(3m_\mathrm{H} - m_\mathrm{e})}\frac{m_\mathrm{H}}{m_\mathrm{e}} \doteq 84.4\,\mathrm{pm}\,.$$

As we can see, the result and the radius of the helium nucleus differ by at least four orders of magnitude, so the particles won't collide.

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Problem FoL.8 ... semicircular analyzer

There is a specific instrument to analyze the distribution of the electrons flying out of the experimental device. It consists of two semicircles of $R = 20\,\mathrm{cm}$ radius, with a magnetic field between them. A half of the cut edge of the semicircle is the detector, while in the second half, there is, in the distance $d = 15\,\mathrm{cm}$ from the center, a crevice through which the electrons are ejected into the space with magnetic field. Assume the electrons are moving at nonrelativistic speed. Derive the ratio η of maximal and minimal velocity of the electrons, measurable with this device.

Aleš wrote down the first thing which came onto his mind.

Lorenz force will act on every particle which passes through the crevice into the analyzing device. The particle falls onto the detector in such manner that the point of the impact will be $2r$ from the crevice where r is a so called Larmor radius. Larmor radius can be derived from the knowledge of the forces acting on the particle. There is actually only one force in this case, the centrifugal force, which is represented by the Lorenz force. Lorenz force depends on the initial velocity of the particle v, its charge q and mass m and also on the magnetic induction B. Lorenz force ($\boldsymbol{F}_\mathrm{L} = q\boldsymbol{v} \times \boldsymbol{B}$) in the smartly chosen coordinate system (in the directions $\boldsymbol{B}$ and $\boldsymbol{v}$) can be written as

$$\frac{mv^2}{r} = qvB\,,$$
$$r = \frac{mv}{qB}\,.$$

If this radius corresponds to half of the distance between the closest point of the crevice

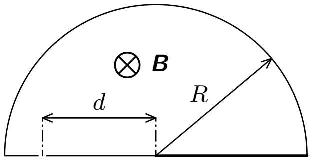

Fig. 1: Analyzator

and the detector, the minimal speed must be $v_\mathrm{min} = dqB/2m$, while in the case of the furthest point, the speed will be maximal $v_\mathrm{max} = (d+R)qB/2m$. Since we are interested in the ratio, we can write:

$$\eta = \frac{v_\mathrm{max}}{v_\mathrm{min}} = \frac{\frac{(d+R)qB}{2m}}{\frac{dqB}{2m}} = \frac{d+R}{d}\,.$$

Plugging in the numbers, we arrive at $\eta \doteq 2.33$.

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Problem FoL.9 ... upgrade

Lately, CERN upgraded their super collider. They increased the energy of the accelerated protons from $3.5\,\mathrm{TeV}$ *to* $7\,\mathrm{TeV}$*. We would like to know how it affected the velocity of the protons. Invariant mass of the proton is* $938\,\mathrm{MeV}/c^2$*.*

Jakub was a bit interested by this question.

We will use the equation $E = \gamma mc^2$, which will help us derive Lorentz factor in both cases (which is approximately 7463 for $7\,\mathrm{TeV}$ and 3731 for $3.5\,\mathrm{TeV}$). Then we can rewrite the equations

$$\gamma = \sqrt{\frac{1}{1-\frac{v^2}{c^2}}}$$

as

$$v = \frac{\sqrt{\gamma^2 - 1}}{\gamma} c\,.$$

Since we are interested in the difference between the velocities, we simply deduct one value from the other

$$\Delta v = \left(\frac{\sqrt{\gamma_1^2 - 1}}{\gamma_1} - \frac{\sqrt{\gamma_2^2 - 1}}{\gamma_2} \right) c\,.$$

Hence the result is approximately $8.1\,\mathrm{m\cdot s^{-1}}$, hence it's very small.

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Problem FoL.10 ... let there be equality

In the container, there is a mixture of $m_1 = 50\,\mathrm{g}$ of iodide ^{131}I and $m_2 = 20\,\mathrm{g}$ of strontium ^{90}Sr. How long will it take to have the same number of atoms of each element in that container?
Taken from Chemistry Olympics.

Half-life of iodide is $T_1 = 8.02\,\mathrm{d}$ and its amount of substance is $M_1 = 131\,\mathrm{g\cdot mol^{-1}}$, while the values for strontium are $T_2 = 28.8\,\mathrm{y}$ and $M_2 = 90.0\,\mathrm{g\cdot mol^{-1}}$. We want an equal number of atoms of both elements, hence

$$\frac{m_1}{M_1} 2^{-\frac{t}{T_1}} = \frac{m_2}{M_2} 2^{-\frac{t}{T_2}}\,,$$

in other words

$$\frac{m_1 M_2}{m_2 M_1} = 2^{\frac{t}{T_1} - \frac{t}{T_2}}\,,$$

where

$$t = \frac{T_1 T_2}{(T_2 - T_1)\ln 2} \ln \frac{m_1 M_2}{m_2 M_1} \doteq 6.26\,\mathrm{d}\,.$$

Notice that strontium's half-life is much longer than iodide's and the amount of substance of iodide is almost twice as big as of strontium. We can use this information to estimate that the time will be somewhat close to T_1. This can be a quick check just before we hand in the problem.

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Problem FoL.11 ... apple cider

Consider a homogeneous cylinder with both pedestals of mass $M = 100\,\mathrm{g}$ and height $H = 20\,\mathrm{cm}$. The cylinder is hanging in the air so its axis of symmetry is identical with the direction of the gravitational acceleration. At first, the cylinder is full of apple cider of mass $m_0 = 500\,\mathrm{g}$. Let's make a small hole in the bottom of the cylinder, so the cider runs out of it. Compute the height of the surface of the cider for the lowest position of the center of mass of the system. Give the result in cm.
Domča likes fall fruits and their derivatives.

Let's write the distance between the bottom of the cylinder and the center of the mass as y (the coordinate system is positive in the upwards direction). The center of mass of the cylinder is in a constant height $H/2$, the cider's center of mass is in $h/2$, while h is the level of the liquid, which depends on the current mass of the cider, m. Hence if we want to know the center of mass of the whole system, we need to compute an average of the centers of mass, which is

$$y = \frac{MH + mh}{2(m + M)}\,.$$

The mass of the cider can be computed as the product of the volume and the density ϱ, where the volume can be computed from the knowledge of the area of the cylinder's bottom S and the current level of the cider $m = Sh\varrho$. Using this, we can rewrite the first equation as

$$y = \frac{MH + h^2 S\varrho}{2(hS\varrho + M)}\,.$$

Since we are interested in the minimum, we have to compute the derivative of the relation with respect to h. The derivative should be zero (we are looking for a stationary point), hence we will obtain a quadratic equation

$$h^2 S\,\varrho + 2Mh - MH = 0\,.$$

Solving the equation and taking into account that only the positive root makes sense, we arrive to

$$y_{\min} = \frac{MH}{m_0}\left(\sqrt{1 + \frac{m_0}{M}} - 1\right)\,.$$

Plugging in the numbers, we get the level of cider to be 5.8 cm.

Solution without derivatives also exists and we leave it to readers.

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Problem FoL.12 ... Wien's filter

If you ever need to filter particles of a certain velocity from others, you can simply do that using perpendicular magnetic and electric fields. Let's assume both of them are homogeneous and oriented so that the particle flying through them flies in a perpendicular direction to both of them. The forces with which the fields are acting on the particles are anti parallel. What is the velocity the particle must have in order to reach the detector or in other words to pass the velocity filter which is aligned with the original direction of motion of the particle? Electric field intensity is $E = 9 \cdot 10^3\ \mathrm{V{\cdot}m^{-1}}$ *and the magnetic induction is* $B = 3 \cdot 10^{-2}\ \mathrm{T}$.

From the experimental physicist's life.

We want the particle to travel straight, hence the Lorentz force $\boldsymbol{F} = q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B})$ has to be zero. Considering the correct orientation of both fields and the fact that the particle is traveling perpendicular to the magnetic induction, we can express the velocity as $v = E/B$. Plugging in the numbers, we get $v \doteq 3 \cdot 10^5\ \mathrm{m{\cdot}s^{-1}}$.

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Problem FoL.13 ... pseudo-ice tea

Kiki felt like having some ice tea, but she didn't know how to prepare it. So she tried to do it in the following way: she put $m_\mathrm{w} = 250\,\mathrm{g}$ *of water at temperature* $t_\mathrm{w} = 20\,^\circ\mathrm{C}$ *into a kettle along with* $m_i = 350\,\mathrm{g}$ *of ice at temperature* $t_i = 0\,^\circ\mathrm{C}$. *After some time (precisely when the contents of the kettle reached the thermal equilibrium), she figured out that she might want to turn on the kettle. How long will it take to boil the water in the kettle, whose power input and efficiency are known to be* $P = 1.8\,\mathrm{kW}$ *and* $80\,\%$ *respectively?*

Kiki was drinking tea while thinking of problems for the competition.

Before turning on the electric kettle, we have to compute the change in temperature of the water from t_w to t_1 at which the mixture of ice and water reaches equilibrium. Let us assume that not all of the ice will be melted. The mass of the remaining ice is then

$$m'_\mathrm{i} = m_\mathrm{i} - \frac{m_\mathrm{w} c (t_\mathrm{w} - t_1)}{l_\mathrm{i}}\,,$$

where $t_1 = 0^\circ\mathrm{C}$. This gives $m'_\mathrm{i} \doteq 287\,\mathrm{g} > 0\,\mathrm{g}$, hence our assumption was right. After turning on the kettle, the rest of the ice will melt and become liquid water. This water added to the original water is the total amount of water we want to heat up from temperature $t_1 = 0^\circ\mathrm{C}$ to $t_2 = 100\,^\circ\mathrm{C}$. The specific latent heat of fusion for ice is $l_\mathrm{f} = 334\,\mathrm{kJ{\cdot}kg^{-1}}$ and the specific heat capacity of water is $c = 4.18\,\mathrm{kJ{\cdot}kg^{-1}}$. We can also write $Pt\eta = Q$, where t is the time and Q is the heat energy necessary for the water to start boiling. Hence the time needed can be expressed as

$$t = \frac{Q}{P\eta}\,,$$

$$t = \frac{m'_\mathrm{i} + (m_\mathrm{i} + m_\mathrm{w}) c (t_2 - t_1)}{P\eta} = \frac{m_\mathrm{i} l_\mathrm{f} - m_\mathrm{w} c (t_\mathrm{w} - t_1) + (m_\mathrm{i} + m_\mathrm{w}) c (t_2 - t_1)}{P\eta}\,.$$

Plugging in the numbers, we get $t \doteq 241\,\mathrm{s}$ ($t \doteq 4\,\mathrm{min}\,1\,\mathrm{s}$).

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Problem FoL.14 ... climbing acetone

Find the elevation of acetone at temperature $20\,^\circ\mathrm{C}$ *in a capillary with diameter* $0.6\,\mathrm{mm}$. *Surface tension of acetone is* $0.023\,4\,\mathrm{N{\cdot}m^{-1}}$. *Your result should be stated in centimetres.*

Kiki remembered a set of problems from physical chemistry.

Force due to surface tension causing elevation is $F = \sigma l$, where σ is surface tension and l is the length of the surface rim. Elevation force is in equilibrium with gravitational force $F_\mathrm{g} = mg$, where m is the mass of the column of liquid. The length of the rim can be written as $l = 2\pi r$, where r is the radius of the capillary. The mass m can be expressed as $m = \varrho V = \varrho \pi 2h$, where ϱ is the density of the acetone and h is the height of the column of liquid we are trying to find. Equating the forces, we have

$$h = \frac{2\sigma}{r\varrho g}\,.$$

Plugging in the numbers yields $h = 2.0\,\mathrm{cm}$. The density of acetone can be found on the Internet, it is approximately $\varrho = 790\,\mathrm{kg{\cdot}m^{-3}}$.

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Problem FoL.15 ... the round one

Imagine a small glass ball with the property that it focuses parallel rays of light incident perpendicularly on its surface onto its back side. What is the refractive index of the glass the ball is made of? Work in paraxial approximation. The result should be expressed as a multiple of the refractive index of ball's surroundings.

Zdeněk found a very suspicious ball in his drawer.

Let us draw the ray diagram of light passing through the glass ball and denote the distances and angles as in the figure. From the isosceles triangle with two equal sides R we can write

$$\alpha = 180^\circ - (180^\circ - 2\beta) = 2\beta\,.$$

Using paraxial approximation $b \ll R$, we can write Snell's law as $n_1\alpha = n_2\beta$, where n_1 is the refractive index of the ball's surroundings and n_2 is the refractive index of the ball itself. Substituting for α yields in $n_2 = 2n_1$, hence the refractive index of the sphere must be twice as big as the refractive index of the environment.

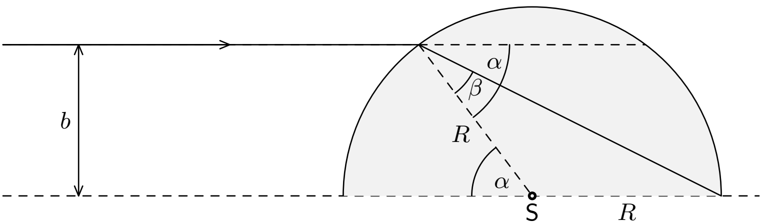

Fig. 2: Light deflection in sphere

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Problem FoL.16 ... the bulb and the capacitor

When Oliver was in New York, he bought a GE 31546 60A1 P VRS ES 110 120V BE type light bulb. When he came back to Czech republic, he didn't really want to throw it away, so in order for the bulb to work in Czech republic he had to connect it in series with an ideal capacitor. What should be the capacitance of this capacitor so that the correct nominal value of the voltage across the light bulb is obtained? The alternating current in European distribution network has frequency $50\,\mathrm{Hz}$ *and the household outlet voltage in Czech Republic is* $230\,\mathrm{V}$*. Do not take into account the internal resistance of the source. The answer should be stated in* μF*.*

Jimmy was inspired by a friend of his, who bought a wrong bulb.

Since the capacitor and the light bulb are connected in series, we can assume that per unit time, there is same charge passing through both of the them, so the current has to be the same through both the bulb and the capacitor. This current can be obtained from the characteristics of the bulb as $I = P/U = 0.5\,\mathrm{A}$. The capacitor can be described by

$$X_\mathrm{C} = \frac{1}{\omega C} \Rightarrow C = \frac{1}{2\pi\nu X_\mathrm{C}} = \frac{I}{2\pi\nu U_\mathrm{C}}\,.$$

It remains to determine the voltage across the capacitor. Current leads the voltage so we can write $U^2 = U_C^2 + U_\mathrm{B}^2$, since the bulb acts as a resistance and so there is no phase difference between the current and the voltage across the bulb. From this we get our final expression

$$C = \frac{P}{2\pi\nu U_\mathrm{Ž}\sqrt{U^2 - U_\mathrm{Ž}^2}} \doteq 8.11\,\mu\mathrm{F}\,.$$

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Problem FoL.17 ... the rod on the wires

Consider a rod of mass $m = 97\,\mathrm{kg}$ suspended on two steel wires Q and W with equal radii $r = 1.3\,\mathrm{mm}$ and elastic moduli $E = 210\cdot 10^9\,\mathrm{Pa}$ in such a way that the rod is parallel to the horizontal, as we can see in the figure. Wire Q was originally (before we used it to suspend the rod) $l_0 = 2.7\,\mathrm{m}$ long while wire W was by $\Delta l = 2\,\mathrm{mm}$ longer. Let us denote the horizontal distances of the wires from rod's center of mass by d_Q, d_W, as we did in the figure. What is the ratio d_Q/d_W? The acceleration due to gravity is $g = 10\,\mathrm{m\cdot s^{-2}}$. The result should be stated up to two significant figures. Assume that the radii of the wires stay unchanged.

Domča was playing with ropes made of construction steel.

The wire Q behaves according to the Hook's law, so the force acting on it must satisfy

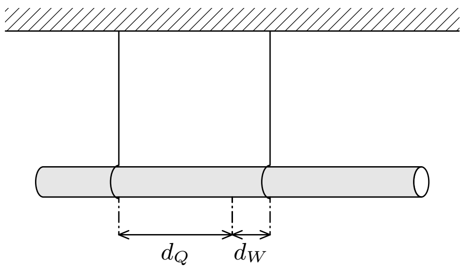

Fig. 3: Rod on wires

$F_Q = SE\Delta l/l_0 = \pi r^2 E\Delta l/l_0$. This force, along with the force F_W with which the rod acts on the wire W, must compensate the gravitational force mg, so we have $F_W = mg - F_Q$. Since the rod is at rest and there is no torsion, the torques acting on it must be in equilibrium – relative to the centre of mass, we can write

$$F_\mathrm{Q}\cdot d_\mathrm{Q} = F_\mathrm{W}\cdot d_\mathrm{W}\,.$$

Now, the ratio can be written easily as

$$\frac{d_{\mathrm{Q}}}{d_{\mathrm{W}}} = \frac{F_{\mathrm{W}}}{F_{\mathrm{Q}}} = \frac{mg - \pi r^2 E \Delta l / l_0}{\pi r^2 E \Delta l / l_0}\,.$$

Plugging in the numbers, the ratio of the lengths is 0.17.

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Problem FoL.18 ... totem

In one of the Plitvička lakes, there is a place where the lake is $2\,\mathrm{m}$ *deep and a vertical totem sticks out of the water reaching to the height of one meter above the water level. The rays coming from the setting Sun located at the altitude of* 30° *above the horizon are incident on the totem. How long is the shadow (in meters) cast by the totem onto the bottom of the lake?*

Dominika was watching Karl May's classics.

The length of the shadow cast on the water level is s_{h}, while its length cast on the bottom of the lake is s_{d}. Refractive indices of water and air are $n = 1.33$ and $n' = 1$ respectively. We know that the angle of incidence of the light is 60°, so the light casts a shadow on the water level with a length of $s_{\mathrm{h}} = (1\,\mathrm{m})/(\mathrm{tg}\,30^\circ)$. The light is then refracted according to Snell's law

$$n' \sin 60^\circ = n \sin \alpha\,,$$

where *alpha* is the angle of refraction of the rays entering the water. The shadow on the bottom of the lake will be longer due to these refracted rays, so

$$s_{\mathrm{d}} = s_{\mathrm{h}} + 2\,\mathrm{m} \cdot \mathrm{tg}\left(\arcsin\left(\frac{1}{n}\sin 60^\circ\right)\right) = 3.45\,\mathrm{m}\,.$$

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Problem FoL.19 ... solar mania

The Sun shines onto the Discworld at an angle of 39° *measured relative to its horizon. In such a situation the illuminance cast on its surface is* $E_1 = 80 \cdot 10^3\,\mathrm{lx}$. *Find the illuminance cast on the surface of the Discworld when the Sun is only* 30° *above the horizon.*

f(Aleš) was lacking light.

For the illumination cast on a surface we have

$$E = \frac{I}{h^2}\cos\alpha = \frac{I}{h^2}\cos\left(\frac{\pi}{2} - \beta\right)\,,$$

where α is the angle of incidence of the incoming rays and β is the angle which the rays make with the horizon. For the ratio of illuminances for both angles of incidence we can write

$$\frac{E_2}{E_1} = \frac{\frac{I}{h^2}\cos\left(\frac{\pi}{2} - \beta_2\right)}{\frac{I}{h^2}\cos\left(\frac{\pi}{2} - \beta_1\right)}\,.$$

From this equation, we can write

$$E_2 = E_1 \frac{\cos\left(\frac{\pi}{2} - \beta_2\right)}{\cos\left(\frac{\pi}{2} - \beta_1\right)}\,.$$

Plugging in the numbers, we get $E_2 \doteq 63\,600\,\mathrm{lx}$.

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Problem FoL.20 ... a decaying one

Imagine we have a $20.0\,\mathrm{g}$ *sample of an unknown radioactive element whose nuclei are known to contain 232 nucleons. In the sample,* $2.12 \cdot 10^{11}$ *decay events per minute are observed to occur. Find the half-life of the element. The product of the decay is assumed to be stable.*
f(Aleš) having fun during a lecture on nuclear physics.

For the number of particles we can write

$$N_0 = \frac{m}{Am_\mathrm{u}}\,,$$

where A is the mass number, m_u atomic mass unit and m is the mass of the radioactive material considered. The number of decay events is given by

$$N' = N_0 \lambda t\,,$$

where t is time and λ is the decay constant, which can be expressed in terms of the above defined quantities as

$$\lambda = \frac{N' A m_\mathrm{u}}{mt}\,.$$

For the half-life T we can derive

$$T = \frac{\ln 2}{\lambda}\,.$$

Substituting for λ, we get

$$T = \frac{mt \ln 2}{N' A m_u}\,.$$

Plugging in the given numerical values and taking $m_\mathrm{u} \doteq 1.66 \cdot 10^{-27}\,\mathrm{kg}$, we obtain the half-life of $T \doteq 1.02 \cdot 10^{13}\,\mathrm{s}$.

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Problem FoL.21 ... thermo-sphere

It is widely known that ordinary light bulbs emit a substantially larger portion of their radiation in infrared than they do in visible. Imagine our heating went on strike and we wish, using our light bulb, to warm up our hands from the temperature $T_1 = 15\,^\circ\mathrm{C}$ *to* $T_2 = 35\,^\circ\mathrm{C}$*. We go on and cover the bulb entirely by our hands, thus exploiting all the thermal power emitted by the incandescent tungsten filament. Find the time needed to bring our frozen hands to the required temperature given the temperature of the tungsten filament is known to be* $T_W = 3\,000\,\mathrm{K}$*, its* *length $l = 10^{-1}$ m and its diameter $d = 10^{-4}$ m. We estimate the mass and the heat capacity of our hands as $m = 1\,\mathrm{kg}$ and $c = 3\,000\,\mathrm{J\cdot K^{-1}\cdot kg^{-1}}$ respectively.*

Mirek and his striking heating.

The total amount of heat necessary to warm our hands is given by $Q = mc(T_2 - T_1)$. The total radiative power of the light bulb can be inferred from the Stefan-Boltzmann law as $P = \sigma\pi dlT_\mathrm{W}^4$. The time needed is then given by

$$t = \frac{Q}{P} = \frac{mc(T_2 - T_1)}{\sigma\pi dlT_\mathrm{W}^4}\,.$$

Substituting the numerical values, we obtain $416\,\mathrm{s} = 6\,\mathrm{min}\,56\,\mathrm{s}$. Hence we observe that the light bulb can be used as a small heater.

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Problem FoL.22 ... analogue hydrometer

Due to its higher density, cold water stays close to the bottom of a rectangular vessel which is filled up to the height of $h = 30\,\mathrm{cm}$. We assume that the density of water in the vessel grows linearly with increasing depth – at the water level, the density is equal to $\varrho_l = 996\,\mathrm{kg\cdot m^{-3}}$, while the density ϱ_b at the bottom of the vessel is unknown. Determine this density using the fact, that a homogeneous rod with density $\varrho_\mathrm{r} = 997\,\mathrm{kg\cdot m^{-3}}$ and length h immersed in the water and fixed by one of its ends at the water level makes an angle of $\varphi = 60^\circ$ with the vertical.

Mirek was thinking about alternative measuring instruments.

At depth x, the density is determined by

$$\varrho(x) = \varrho_\mathrm{l} + \frac{x}{h}\left(\varrho_\mathrm{b} - \varrho_\mathrm{l}\right).$$

For the rod to be at equilibrium, it is required that the total torque acting on the rod is zero, i.e.

$$\int \mathrm{d}M = 0\,.$$

We can express the elementary torque acting on an infinitely small section of the rod as $\mathrm{d}M = x(\mathrm{d}F_\mathrm{vz} - \mathrm{d}F_\mathrm{g})$ where the elementary forces $\mathrm{d}F_\mathrm{vz}$ and $\mathrm{d}F_\mathrm{g}$ are given by

$$\mathrm{d}F_\mathrm{vz} = \frac{mg\varrho(x)}{\varrho_\mathrm{r}h}\mathrm{d}x\,,$$
$$\mathrm{d}F_\mathrm{g} = \frac{mg}{h}\mathrm{d}x\,,$$

where m is the mass of the rod. Integrating from 0 to $h\cos\varphi$ (where our x-axis is directed vertically downwards and $h\cos\varphi$ is the x-coordinate of the lower end of the rod), we have

$$\int_0^{h\cos\varphi} x\left(\frac{mg}{\varrho_\mathrm{r}l}\left(\varrho_\mathrm{l} + \frac{x}{h}(\varrho_\mathrm{b} - \varrho_\mathrm{l})\right) - \frac{mg}{l}\right)\mathrm{d}x = 0\,,$$
$$\frac{\varrho_\mathrm{l}}{2\varrho_\mathrm{r}l} + \frac{h\cos\varphi}{3\varrho_\mathrm{r}hl}\left(\varrho_\mathrm{b} - \varrho_\mathrm{l}\right) - \frac{1}{2l} = 0\,.$$

It remains to express ϱ_b and substitute the numerical values. Eventually, we obtain

$$\varrho_\mathrm{b} = \frac{3}{2\cos\varphi}\left(\varrho_\mathrm{r} - \varrho_\mathrm{l}\right) + \varrho_\mathrm{l} \doteq 999\,\mathrm{kg{\cdot}m^{-3}}\,.$$

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Problem FoL.23 ... pressurized box

Assume we have $n = 1\,\mathrm{mol}$ of carbon dioxide (CO_2) in a closed vessel with volume $V = 1\,\mathrm{l}$. The vessel is in thermal equilibrium with its surroundings at temperature $T = 297\,\mathrm{K}$. How does the estimate of pressure in the vessel based on the ideal gas law (denote p_{id}) differ from the the estimate based on the van der Waals equation (1) of state for non-ideal fluid (denote p_Waals)?

$$\left(p_\mathrm{Waals} + \frac{n^2 a}{V^2}\right)(V - nb) = nRT \qquad (1)$$

Determine $\left(p_{id} - p_\mathrm{Waals}\right)/p_\mathrm{id}$. Use the following values of a, b for CO_2:

$$a = 0.3653\,\mathrm{Pa{\cdot}m^6{\cdot}mol^{-2}}\,,$$
$$b = 4.280\cdot 10^{-5}\,\mathrm{m^3{\cdot}mol^{-1}}\,.$$

The molar gas constant is $R = 8.31\,\mathrm{J{\cdot}K^{-1}{\cdot}mol^{-1}}$.

Karel wanted to mention van der Waals gas.

Let us express the estimates of pressure from both equations

$$p_\mathrm{id} = \frac{nRT}{V}\,,$$
$$p_\mathrm{Waals} = \frac{nRT}{V - nb} - \frac{n^2 a}{V^2}\,.$$

It remains to calculate the ratio

$$\frac{p_\mathrm{id} - p_\mathrm{Waals}}{p_\mathrm{id}} = 1 - \frac{V}{V - nb} + \frac{na}{RTV} \doteq 10.3\,\%\,.$$

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Problem FoL.24 ... world champion in high jump

Brian Griffin (a dog) has one spring mounted to each of his hind legs. Each of these springs has an unstretched length of $l = 0.5\,\mathrm{m}$ and spring constant $k = 3\cdot 10^5\,\mathrm{kg{\cdot}s^{-2}}$. The dog then jumps to a vertical height of $h = 10\,\mathrm{m}$ and when he falls down, the springs become hooked into the ground so the dog starts oscillating. What is the amplitude of the undamped oscillations for a dog weighing $m = 500\,\mathrm{kg}$? Assume gravitational accleration $g = 9.81\,\mathrm{m{\cdot}s^{-2}}$.

Mirek envied Brian his funny means of transport.

At the top of his trajectory, the dog has potential energy $E_1 = mgh$. After the impact, the springs absorb energy $E_\mathrm{p} = \frac{1}{2}(k+k)y^2$ and the dog retains energy $E_2 = mg(l-y)$ where y is the maximum compression of each spring. Using the law of conservation of mechanical energy, we obtain a quadratic equation for y

$$ky^2 - mgy - mg(h-l) = 0\,.$$

Using the given numerical values, its positive root is found to be $y \doteq 0.402\,\mathrm{m}$. However, this is not the amplitude. To obtain the amplitude, we have to subtract the compression of springs at equilibrium $y_0 = mg/2k \doteq 0.008\,\mathrm{m}$. Hence the amplitude is $y_\mathrm{a} \doteq 0.394\,\mathrm{m}$.

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Problem FoL.25 ... pulleys

Calculate the vertical acceleration of the weight with mass m_1 in the pulley system depicted in the figure, assuming that the system is initially at rest. The masses of individual weights are $m_1 = 400\,\mathrm{g}$, $m_2 = 200\,\mathrm{g}$, $m_3 = 100\,\mathrm{g}$. Use $g = 10\,\mathrm{m{\cdot}s^{-2}}$ as the value for the acceleration due to gravity. Pulleys and strings are weightless and friction can be neglected.

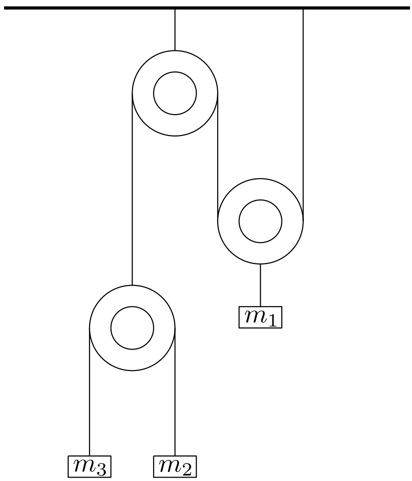

Fig. 4: Pulleys machine

Mirek was amazed at simple machines.

Let us denote by T_2 the tension in the string joining weights 2 and 3. Similarly, let T_1 be the tension in the string attached to the leftmost free pulley, wound around the fixed pulley and the rightmost free pulley. Accelerations of the individual weights are denoted by a_1, a_2 and a_3,

a_4 is the acceleration of the leftmost pulley. Assume that all the above defined accelerations are directed downwards. Then we can describe the system with following equations

$$\begin{aligned} m_1a_1 &= m_1g - 2T_1\,, \\ m_2a_2 &= m_2g - T_2\,, \\ m_3a_3 &= m_3g - T_2\,, \\ T_1 &= 2T_2\,, \\ 2a_1 &= -a_4\,. \end{aligned}$$

We define the weights' accelerations relative to the leftmost pulley as a'_2 and a'_3 respectively. For these we can write $a'_2 = -a'_3$, $a_2 = a'_2 + a_4$ and $a_3 = -a'_2 + a_4$, which we use to derive the sixth equation

$$a_1 = -\frac{1}{4}(a_2 + a_3)\,.$$

We use the last three equations to substitute into the first three, so we obtain

$$\begin{aligned} -\frac{1}{4}m_1(a_2 + a_3) &= m_1g - 4T_2\,, \\ m_2a_2 &= m_2g - T_2\,, \\ m_3a_3 &= m_3g - T_2. \end{aligned}$$

Adding the second equation to the third and comparing the result with the first, we get

$$\frac{16T_2}{m_1} - 4g = 2g - T_2\left(\frac{1}{m_2} + \frac{1}{m_3}\right)\,.$$

Solving this for T_2 yields

$$T_2 = 6g\frac{m_1m_2m_3}{m_1m_2 + m_1m_3 + 16m_2m_3}\,.$$

Now we substitute for T_2 into the equations relating accelerations $a_2 = g - T_2/m_2$, $a_3 = g - T_2/m_3$ and using fifth and sixth relation, we get

$$a_1 = g\frac{m_1m_2 + m_1m_3 - 8m_2m_3}{m_1m_2 + m_1m_3 + 4m_2m_3} = -2\,\mathrm{m{\cdot}s^{-2}}\,.$$

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Problem FoL.26 ... boxed light

How many photons of blue light does it take to achieve pressure $p = 1\,\mathrm{bar} = 10^5\,\mathrm{Pa}$ in an empty cube-shaped box? The wavelength of the photons is $\lambda = 450\,\mathrm{nm}$ and the edge length of the cube is $a = 10\,\mathrm{cm}$. The interior of the cube is perfectly reflective.

Thermodynamics class acted upon Karel.

We will use the same reasoning as when deriving the pressure of an ordinary gas. A wall of area S is being hit by $Nct/(6V)$ photons over a time period t, where N is the number of particles, V is the volume of the box, c is the speed of light and $1/6$ is the particles' effective direction factor – only one sixth of them is moving in the direction of positive x-axis. When a photon hits the wall, its momentum is changed by $2h/\lambda$. The change in total momentum of all photons reflected at the area S during time period t is

$$\Delta p_{\mathrm{m}} = \frac{hcNtS}{3\lambda V}\,.$$

Force is defined as the rate of change of momentum while pressure is a force per unit area acting perpendicularly to a surface. Thus

$$p = \frac{hcN}{3\lambda V}\,.$$

We need to find the number of particles

$$N = \frac{3p\lambda V}{hc} \doteq 6.79 \cdot 10^{20}\,.$$

With this number of particles, the radiation energy density in the box is about $3 \cdot 10^5\,\mathrm{J{\cdot}m^{-3}}$, which is roughly six orders of magnitude greater than the radiation energy density at the surface of the Sun.

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Problem FoL.27 ... enlightened carriage

A carriage of mass $m = 100\,\mathrm{g}$ can move along rails without friction. Assume there is a vertical mirror attached to one side of the carriage. We then focus all the light emitted by a light bulb with power $P = 60\,\mathrm{W}$ into a ray which is incident perpendicularly onto the mirror. Assuming that the carriage was initially at rest, how long will it take the carriage to travel a distance $l = 1\,\mathrm{m}$ (in seconds)? You can assume that the mirror is perfectly reflective and that the whole power of the light bulb transferred into radiation.

Jakub wanted a contactless turbo-propulsion.

The light which is reflected from the mirror possesses certain momentum. After the reflection, the sign of the momentum will change. Since the momentum is conserved, the carriage must have gained some momentum in the process. Momentum of a photon with wavelength λ is $p = h/\lambda$, where h is the Planck's constant. Force acting upon the carriage can be computed by dividing the change in momentum by the time period over which the change took place, i.e.

$$F = \frac{\Delta p}{\Delta t}\,.$$

The change in momentum of a photon is is equal to twice its momentum itself since, upon reflection, the photon changes the direction in which it moves. The relation between wavelength λ and the frequency f of a photon is $\lambda = c/f$. Substituting for λ from this, we get

$$F = \frac{2hf}{c\Delta t}\,.$$

Energy E of a photon is $E = hf$. Substituting for E from this, we arrive at

$$F = \frac{2E}{c\Delta t}\,.$$

This energy is the energy of a photon (photons) reflected over a time period of Δt and it also corresponds to the energy output of the light bulb over the same period. Dividing the energy by the time period we obtain the power of the bulb P

$$F = \frac{2P}{c} \,.$$

This constant force causes constant acceleration

$$a = \frac{F}{m} = \frac{2P}{cm}$$

of the carriage. Thus the time t needed for the carriage to cover distance l satisfies $l = at^2/2$. Expressing t from this and plugging in the numbers, we get

$$t = \sqrt{\frac{lcm}{P}} \doteq 707\,\mathrm{s}\,.$$

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Problem FoL.28 ... Earth-cylinder

Assume that the Earth turned into an infinite cylinder whose radius and density are the same as those of our real Earth with the distance to the Moon also remaining unchanged. What will be the speed of the Moon (which remains spherical) in its orbit around the cylinder? Assume that the Earth's radius and density are $6\,378\,\mathrm{km}$ *and* $5\,515\,\mathrm{kg{\cdot}m^{-3}}$ *respectively. Also consider the Moon's orbit around the Earth to be circular.*

Tomáš Bárta imagining alternative universes.

Let us denote the gravitational field intensity by $\boldsymbol{K}$. We will treat the gravitational field in much the same way as we usually do with electromagnetic field. Gauss' law can be written as

$$\oint_S \boldsymbol{K} \cdot \mathrm{d}\boldsymbol{S} = 4\pi GM \,.$$

We choose a cylinder with radius r and length l as our gaussian surface. The mass enclosed by such a cylinder is $\pi R_\mathrm{E}^2 l \varrho_\mathrm{E}$. Rearranging the above expression for the flux of the gravitational field through the cylinder, we get

$$2\pi r l K(r) = 4\pi G \cdot \pi R_\mathrm{E}^2 l \varrho_\mathrm{E} \,,$$
$$K(r) = \frac{2}{r} G\pi \varrho_E R_\mathrm{E}^2 \,.$$

Finally we equate the magnitudes of centripetal acceleration and the gravitational field intensity and we obtain

$$\frac{v^2}{r} = \frac{2}{r} G\pi \varrho_\mathrm{E} R_\mathrm{E}^2 \,,$$
$$v = R_\mathrm{E} \sqrt{2\pi \varrho_\mathrm{E} G} \doteq 9\,700\,\mathrm{m{\cdot}s^{-1}} \,.$$

Therefore we have reached an interesting conclusion that the speed of the Moon in its orbit around cylindrical Earth does not depend on the distance of the Moon from the axis of the cylinder.

Problem FoL.29 ... pass me the hammer

An astronaut dropped a tool bag during one of his spacewalks, giving it an impulse. The bag then started receding from the spaceship along a straight line until it reached the distance $l = 180\,\mathrm{m}$ *from the ship. At this distance, the speed of the bag relative to the spaceship was zero. How many days did it take the bag to get to that spot? The mass of the bag and the mass of the spaceship are known to be* $m_1 = 50\,\mathrm{kg}$ *and* $m_2 = 500\,\mathrm{kg}$ *respectively. You can neglect any effects due to gravity of other bodies.* *Lukáš remembered the infamous ISS bag.*

The motion of our ill-fated bag will obey Kepler's laws of planetary motion. Note that straight line is only a special (degenerate) case of an ellipse, where the numerical eccentricity is equal to 1. Hence the distance l is twice the semi-major axis of such an ellipse. We see that it took the bag half of its orbit to reach the described point (apoapsis). Using third Kepler's law we have

$$\frac{\left(\frac{l}{2}\right)^3}{(2t)^2} = \frac{G(m_1+m_2)}{4\pi^2}\,,$$

$$t = \sqrt{\frac{\pi^2 l^3}{8G(m_1+m_2)}}\,.$$

Plugging in the numbers, we get $t \doteq 162\,\mathrm{days}$.

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Problem FoL.30 ... a messy one

A chemist was planning to prepare $100\,\mathrm{ml}$ *of potassium permanganate solution with concentration* $c_1 = 0.000\,5\,\mathrm{mol\cdot dm^{-3}}$*, so he placed the required amount of potassium permanganate into a measuring flask, poured in distilled water and mixed the solution carefully. But then an accident happened and he spilled a part of the solution. He was very lazy and since nobody else saw him, he just topped the solution up to the original volume with distilled water and pretended that nothing happened. Then he started to feel bad about the whole incident, so he took a sample of his solution into a* $1\,\mathrm{cm}$ *long cuvette and put it into a spectrophotometer. From the subsequent measurement he found that having passed through the sample, the intensity of monochromatic light of wavelength* $526\,\mathrm{nm}$ *dropped by* $90\,\%$ *compared to its original value. What was the volume of the solution the chemist spilled assuming that his measurements were precise? Molar absorption coefficient of potassium permanganate for the above given wavelength is* $2\,440\,\mathrm{cm^2\cdot mmol^{-1}}$*. You should give your answer in millilitres.*
Kiki will once become a real pharmacist.

In this problem, we will make use of Lambert–Beer law $A = \varepsilon c l$, where A is the absorbance, for which we can write $A = \log\left(I_0/I\right)$, where I_0 is the intensity of light before it passes through the sample, I is the intensity of light after it passes through the sample, c is the concentration of the given constituent in the sample, l is the length of the cuvette and ε is the molar absorption coefficient. Knowing this, we can compute the present concentration of potassium permanganate in the sample as

$$c_2 = \frac{\log \frac{I_0}{I}}{\varepsilon l}\,.$$

Now we can compute the amount of potassium permanganate in the solution as $m = nM = cVM$, where $M = 158\,\mathrm{g{\cdot}mol^{-1}}$ is the molar mass of potassium permanganate. Plugging in the numbers, we would obtain numerical values for m_1 (100%) and m_2 (x%). Now we can compute $(1 - x/100) \cdot 100\,\mathrm{ml}$, which will give us the volume of solution spilled (remember, the solution was perfectly mixed so we are allowed to use direct proportion), which corresponds to a volume of 18 ml.

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Problem FoL.31 ... valuing the decays

An isotope of gold ^{173}*Au has a half-life of* $T_{\mathrm{Au}} = 59.0\,\mathrm{ms}$. *It decays into an iridium isotope* ^{169}*Ir by emitting an alpha particle. This iridium isotope has a half-life of* $T_{\mathrm{Ir}} = 0.400\,\mathrm{s}$ *and decays into* ^{165}*Re. Initially we had a pure sample of gold isotope* ^{173}*Au. Find the time at which the amount of gold in our sample will be the same as the amount of iridium. You can assume that masses of isotopes are directly proportional to their nucleon numbers.* *Karel was thinking hard.*

Since we have modern technology (such as a computer) at our disposal, we can make use of spreadsheet software like Excel or Numbers. Since the theory of multiple decay is considered to be at university level, we will assume that the solution was attempted using numerical simulations and we will try to reach it here in the same way. Our numerical simulations was created in MS Excel 2007, but any other software or programming language could have been used just as well to complete the task. The simulation can be found in a file published on our website. We used Euler's method, which is the most primitive one, but the easiest one for implementation. Initially we only had gold ^{173}Au, with its maximal initial mass, $m_{\mathrm{Au}}(0)$ in whose multiples any subsequent result will be stated. The time is sampled by 0.01 ms and is stored in column A. We are computing the mass loss in every time step and store it in column D. It is given by

$$\Delta m_{\mathrm{Au}}(t + \Delta t) = m_{\mathrm{Au}}(t) - m_{\mathrm{Au}}(t + \Delta t) = m_{\mathrm{Au}}(t) \cdot \left(1 - 2^{-\frac{\Delta t}{T_{\mathrm{Au}}}}\right)\,.$$

The instantaneous mass of gold is computed as a difference between the initial mass and the mass lost. It is written in column C. Column E then represents the gain of mass of Iridium in the sample, which will be 169/173 times the mass of the gold lost. This fraction was introduced so that we took into account the loss of mass resulting from the emission of an α-particle during the decay. Column F is used to store the instantaneous mass of Iridium 169, which is calculated as a sum of the original value for this mass plus the gain minus the loss of mass due a the further decay to ^{165}Re. This further loss is computed in column G. Column H, which contains the ratio of instantaneous masses of iridium and gold, was introduced in order to simplify our search for the moment, when the amount of gold and iridium were the same. Hence we are looking for the moment, when the value of this ratio reaches 1, which happens between the times 0.062 64 s and 0.062 65 s.

The task can be also solved analytically. It is more time consuming, but also more precise. Let us start with the following two ordinary differential equations

$$\frac{\mathrm{d}N_{\mathrm{Au}}}{\mathrm{d}t} = -\lambda_{\mathrm{Au}} N_{\mathrm{Au}}\,,$$
$$\frac{\mathrm{d}N_{\mathrm{Ir}}}{\mathrm{d}t} = -\lambda_{\mathrm{Ir}} N_{\mathrm{Ir}} - \frac{\mathrm{d}N_{\mathrm{Au}}}{\mathrm{d}t} = -\lambda_{\mathrm{Ir}} N_{\mathrm{Ir}} + \lambda_{\mathrm{Au}} N_{\mathrm{Au}}\,,$$

where $\lambda_{\mathrm{Au}} = \ln 2 T_{\mathrm{Au}}^{-1}$ and $\lambda_{\mathrm{Ir}} = \ln 2 T_{\mathrm{Ir}}^{-1}$. The solution of the first equation is trivial and we can directly substitute it in the second equation which yields

$$\frac{\mathrm{d}N_{\mathrm{Ir}}}{\mathrm{d}t} + \lambda_{\mathrm{Ir}} N_{\mathrm{Ir}} = \lambda_{\mathrm{Au}} N_{\mathrm{Au0}} \mathrm{e}^{-\lambda_{\mathrm{Au}} t}\,,$$

where N_{Au0} is the initial mass of gold. We will multiply the new equation by $\mathrm{e}^{\lambda_{\mathrm{Ir}} t}$ and rearrange so that we get

$$\frac{\mathrm{d}N_{\mathrm{Ir}}}{\mathrm{d}t} \mathrm{e}^{\lambda_{\mathrm{Ir}} t} + \lambda_{\mathrm{Ir}} N_{\mathrm{Ir}} \mathrm{e}^{\lambda_{\mathrm{Ir}} t} = \lambda_{\mathrm{Au}} N_{\mathrm{Au0}} \mathrm{e}^{(\lambda_{\mathrm{Ir}} - \lambda_{\mathrm{Au}}) t}\,,$$

where we make use of the property of exponential that $\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^{\lambda_{\mathrm{Ir}} t}\right) = \lambda_{\mathrm{Ir}} \mathrm{e}^{\lambda_{\mathrm{Ir}} t}$, which together with Leibnitz rule yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(N_{\mathrm{Ir}} e^{\lambda_{\mathrm{Ir}} t}\right) = \lambda_{\mathrm{Au}} N_{\mathrm{Au0}} \mathrm{e}^{(\lambda_{\mathrm{Ir}} - \lambda_{\mathrm{Au}}) t}\,,$$

which is an easily integrable equation. In the integrated equation

$$N_{\mathrm{Ir}} e^{\lambda_{\mathrm{Ir}} t} = \frac{\lambda_{\mathrm{Au}} N_{\mathrm{Au0}} \mathrm{e}^{(\lambda_{\mathrm{Ir}} - \lambda_{\mathrm{Au}}) t}}{\lambda_{\mathrm{Ir}} - \lambda_{\mathrm{Au}}} + c,$$

we need to determine the integration constant c. Using the initial condition $N_{\mathrm{Ir}}(0) = 0$, we get

$$c = -\frac{\lambda_{\mathrm{Au}} N_{\mathrm{Au0}}}{\lambda_{\mathrm{Ir}} - \lambda_{\mathrm{Au}}}\,.$$

Hence we can express the amount of iridium as it depends on time as

$$N_{\mathrm{Ir}} = \mathrm{e}^{-\lambda_{\mathrm{Ir}} t} \frac{\lambda_{\mathrm{Au}} N_{\mathrm{Au0}} \left(\mathrm{e}^{(\lambda_{\mathrm{Ir}} - \lambda_{\mathrm{Au}}) t} - 1\right)}{\lambda_{\mathrm{Ir}} - \lambda_{\mathrm{Au}}}\,.$$

We are looking for the time when $m_{\mathrm{Au}} = m_{\mathrm{Ir}}$. In other words

$$M_{\mathrm{Au}} N_{\mathrm{Au0}} \mathrm{e}^{-\lambda_{\mathrm{Au}} t} = M_{\mathrm{Ir}} \mathrm{e}^{-\lambda_{\mathrm{Ir}} t} \frac{\lambda_{\mathrm{Au}} N_{\mathrm{Au0}} \left(\mathrm{e}^{(\lambda_{\mathrm{Ir}} - \lambda_{\mathrm{Au}}) t} - 1\right)}{\lambda_{\mathrm{Ir}} - \lambda_{\mathrm{Au}}}\,,$$

where M_{Au}, M_{Ir} are molar masses of gold and iridium. Expressed in terms of their half-lives, this becomes

$$\frac{\ln\left(1 - \frac{M_{\mathrm{Au}}}{M_{\mathrm{Ir}}}\left(\frac{T_{\mathrm{Au}}}{T_{\mathrm{Ir}}}\right)\right)}{\frac{\ln 2}{T_{\mathrm{Au}}} - \frac{\ln 2}{T_2}}\,.$$

Plugging in the numbers, we get $t \doteq 0.062\,64\,\mathrm{s}$. Answers lying in the interval from 0.062 5 s to 0.062 8 s were all considered correct since various values for the time sampling could be chosen.

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Problem FoL.32 ... Flash

After an attack, the superhero Flash does not bother to stop. Instead, he circles around the Earth, and attacks again. Initially, his speed was $v = 0.8c$. During the first attack, he looses half of his momentum. After repeating the run around the Earth with his new speed and engaging in the second attack, he looses half of his momentum again. What is the ratio E_1/E_2 of energies released during the first and the second attack? According to dc.wikia.com, Flash's rest mass is $m_0 = 89\,\mathrm{kg}$. *Mirek watching superhero programmes.*

The energy transferred to the enemy during an attack is equal to the loss of Flash's kinetic energy. Because the relativistic effects are important we use Flash's total energy expressed using the magnitude of four-momentum as $E = \sqrt{m_0^2c^4 + p^2c^2}$. Therefore, the kinetic energy is $E_\mathrm{k} = \sqrt{m_0^2c^4 + p^2c^2} - m_0c^2$. The energy losses during the first and the second attack are then given by

$$E_1 = \sqrt{m_0^2c^4 + p^2c^2} - \sqrt{m_0^2c^4 + p^2c^2/4}\,,$$
$$E_2 = \sqrt{m_0^2c^4 + p^2c^2/4} - \sqrt{m_0^2c^4 + p^2c^2/16}\,.$$

For the momentum p we can substitute $p = \gamma m_0 v = (m_0vc)/\sqrt{c^2 - v^2}$, and after some rearrangements we arrive at the final formula

$$\frac{E_1}{E_2} = \frac{1 - \sqrt{1 - \frac{3}{4}\beta^2}}{\sqrt{1 - \frac{3}{4}\beta^2} - \sqrt{1 - \frac{15}{16}\beta^2}}\,,$$

where $\beta = v/c$. Using $\beta = 0.8$, the wanted ratio is $E_1/E_2 \doteq \pi$.

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Problem FoL.33 ... we need to go deeper

What is the moment of inertia (relative to its axis of symmetry o) of the lamina shown in the figure (the lamina contains the shaded parts only)? The shape is constructed in the following way: given a semicircle, we cut out semicircular holes of it with their radii being half of the radius of the original semicircle. We then insert four four-times smaller semicircles (two into each hole) in these holes, again with semicircular holes cut out of them and in these we then insert smaller semicircles, continuing ad infinitum. The mass of the plate is $m = 7\,\mathrm{kg}$, and the radius of the largest semicircle is $R = 40\,\mathrm{cm}$.

Xellos was thinking about the old competition days.

First, let us calculate the mass of the object in the figure as it depends on R. Note that it is just a semicircle with two shapes cut out of it which are basically the same shape as the one whose mass we are trying to compute only twice as small. The object is two-dimensional which means that if the smaller shapes are twice as small, their mass is four times as small. Denoting the surface density of the object as σ, the mass of the semicircle with radius R is $m_0 = \pi R^2\sigma/2$. The mass of our object must satisfy

$$m = m_0 - \frac{m}{2}\,,$$
$$m = \frac{\pi R^2 \sigma}{3}\,.$$

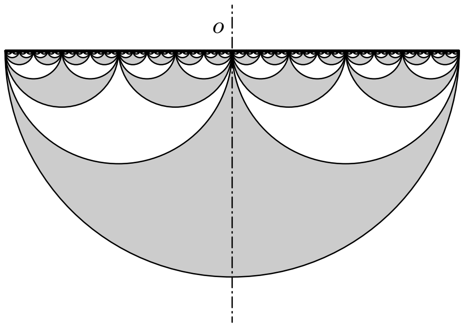

Fig. 5: Lamina

We can use similar reasoning to calculate its moment of inertia. First, the moment of inertia of the semicircle with radius R is $I_0 = m_0 R^2/4$. Let us denote the wanted moment of inertia by I. Halving the radius, I decreases by a factor of 16, as it can be written as $I = KmR^2$, where K is an unknown constant. Using the parallel axis theorem, we find

$$I = I_0 - 2\left(\frac{I}{16} + \frac{m}{4}\left(\frac{R}{2}\right)^2\right) = \frac{\pi R^4\sigma}{8} - \frac{I}{8} - \frac{\pi R^4 \sigma}{24}\,,$$
$$I = \frac{2\pi R^4\sigma}{27} = \frac{2mR^2}{9}\,.$$

Plugging in the numbers, we can find the result to be $0.25\,\mathrm{kg{\cdot}m^2}$.

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Problem FoL.34 ... little skewer

A thin rod of length $2l = 30\,\mathrm{cm}$ is placed into a hemispherical bowl of radius $R = 10\,\mathrm{cm}$. Assuming that the rod is at equilibrium, what is the angle α (in degrees) between the rod and the vertical? *f(Aleš) really liked a rod problem on the internet.*

Let us solve this problem using the principle of virtual work that can be stated as

$$\sum_{i=1}^{N} \boldsymbol{F_i} \cdot \delta \boldsymbol{r_i} = 0\,, \tag{2}$$

where $\boldsymbol{F}_i$ are the real forces acting on the system, and $\delta \boldsymbol{r}_i$ are the displacements allowed by the constraints. There is only one real force acting on the stick being the gravitational force $\boldsymbol{F}_\mathrm{G}$, which acts vertically downwards at the centre of mass of the stick. Let us choose our coordinate system so that the gravitational force acts along y-axis with x-axis perpendicular to the y-axis. Since there are no forces acting in the x direction, we can restrict our analysis to the y-component of motion.

The y coordinate of the center of mass can be written in terms of the angle α as

$$y = R\sin 2\alpha - l\cos\alpha,$$

and so the virtual displacement is

$$\delta y = \frac{\mathrm{d}y}{\mathrm{d}\alpha}\delta\alpha = \left(2R\cos 2\alpha + l\sin\alpha\right)\delta\alpha\,.$$

Equation (2) then states that

$$F\delta y = mg\left(2R\cos\alpha + l\sin\alpha\right)\delta\alpha = 0\,.$$

This equation has two solutions, but only the positive one is physically admissible. It is equal to

$$\sin\alpha = \frac{l}{8R} + \frac{1}{2}\sqrt{\frac{l^2}{16R^2} + 2}\,.$$

Plugging in the numbers we find $\alpha \doteq 67^\circ$.

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Problem FoL.35 ... tesseract

An expedition exploring the ocean floor has found a four-dimensional cube called tesseract or hypercube. Naturally, the explorers decided to investigate its physical properties, so they tried to melt it. They found out that it was made of an isotropic material with a large coefficient of linear thermal expansion which, in addition, was found to grow linearly with increasing temperature. Specifically, $\alpha_a(T_1) = 5\cdot 10^{-4}\,\mathrm{K}^{-1}$ *and* $\alpha_a(T_2) = 2\cdot 10^{-3}\,\mathrm{K}^{-1}$ *for* $T_1 = 300\,\mathrm{K}$ *and* $T_2 = 400\,\mathrm{K}$ *respectively. Find the percentage increase in the 4-volume of the hypercube when heated from* T_1 *to* T_2 *given that at the temperature* T_1*, its edge length is* $a = 10\,\mathrm{cm}$.

Mirek was thinking about the physics in Avengers.

The coefficient of linear thermal expansion is defined by

$$\alpha_a = \frac{1}{a}\frac{\mathrm{d}a}{\mathrm{d}T}$$

Similarly, for a change in volume we define

$$\alpha_V = \frac{1}{V}\frac{\mathrm{d}V}{\mathrm{d}T}\,.$$

The volume of a hypercube is $V = a^4$, and for very small changes in temperature we have

$$V + \mathrm{d}V = (a + \mathrm{d}a)^4 \approx a^4 + 4a^3\mathrm{d}a = V + 4V\frac{\mathrm{d}a}{a}.$$

Using $\mathrm{d}V = \alpha_V a^4 \mathrm{d}T$, $\mathrm{d}a = \alpha_a a \mathrm{d}T$, we arrive at

$$a^4 + a^4 \alpha_V \mathrm{d}T = a^4 + 4a^4 \alpha_a \mathrm{d}T\,,$$

so that $\alpha_V = 4\alpha_a$. The change in volume is obtained by a simple integration

$$\int_{V_1}^{V_2} \frac{\mathrm{d}V}{V} = \int_{T_1}^{T_2} \alpha_V(T)\mathrm{d}T\,,$$

$$\ln \frac{V_2}{V_1} = 2\left(T_2 - T_1\right)\left(\alpha_a(T_2) + \alpha_a(T_1)\right)\,.$$

The percentage increase is then

$$\frac{V_2 - V_1}{V_1} = \left(\mathrm{e}^{2(T_2 - T_1)(\alpha_a(T_2) + \alpha_a(T_1))} - 1\right) \cdot 100\,\% \doteq 64.9\,\%\,.$$

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Problem FoL.36 ... short hands

Mirek wanted to measure the dimensions of his room but all he could use was a 2 metres long folding rule. With each of his hands taking hold of one end of the ruler, he found out that to his surprise he could not straighten the ruler by spreading out his arms. What is the vertical distance between the lowest point of the sagged ruler and the horizontal line joining Mirek's hands? The ruler consists of four identical sections of length $l = 0.5\,\mathrm{m}$ and Mirek's arm span is $L = 1.8\,\mathrm{m}$. You should neglect friction and overlaps between the individual sections of the ruler. *Mirek discovered that height and the arm span are more or less the same.*

Exploiting the symmetry of the problem, it is sufficient to describe the system by angles α, β which the first and second section respectively make with the vertical. The total potential energy of the ruler can be expressed as

$$u(\alpha, \beta) = -2mg\frac{l\cos\alpha}{2} + 2mg\left(l\cos\alpha + \frac{l\cos\beta}{2}\right) = -mgl(3\cos\alpha + \cos\beta)\,,$$

where m is the mass of a single section of the ruler and we set the level of zero potential energy to coincide with the horizontal line connecting Mirek's hands. We would like to minimize potential energy subject to the length of the ruler being held fixed. This constraint can be expressed as

$$f(\alpha, \beta) = 2l(\sin\alpha + \sin\beta) - L = 0\,.$$

Using the method of Lagrange multipliers we see that the angles that minimize potential energy satisfy

$$\frac{\partial U}{\partial \alpha} - \lambda \frac{\partial f}{\partial \alpha} = 0\,,$$
$$\frac{\partial U}{\partial \beta} - \lambda \frac{\partial f}{\partial \beta} = 0\,,$$

where λ is the Lagrange multiplier. Expressing λ from these, equating both expressions obtained and taking derivatives we find

$$3 \operatorname{tg} \alpha = \operatorname{tg} \beta \,.$$

A second equation for α and β follows from the constraint. This is a system of transcendental equations with the physically admissible solution $\alpha \approx 55.6\,^{\circ}$, $\beta \approx 77.13\,^{\circ}$. Using simple geometry, the wanted vertical distance of the lowest point can be expressed as

$$h = l(\cos\alpha + \cos\beta) \,,$$

which gives the result $h = 0.394\,\mathrm{m}$.

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Problem FoL.37 ... resistant Fibonacci

Let us consider a configuration of resistors with identical resistances $R = 1\,\Omega$ built as follows: first we connect two resistors in series, next to them we add two resistors connected in parallel followed by 3 resistors in parallel, then 5, 8, 13, ... resistors in parallel until we get a Fibonacci sequence of resistors. Find the resistance of such a configuration. *Mirek could not resist.*

Let us denote the n^{th} term of Fibonacci sequence by $F(n)$. The resistance of the n^{th} term of the sequence will therefore be $R/F(n)$. Hence the key question is to find an expression for $F(n)$.

The recurrence relation for Fibonacci sequence is as follows

$$F_n = F_{n-1} + F_{n-2}, \quad F_1 = 1, \quad F_2 = 1 \,.$$

Solving the auxilliary equation $t^2 = t + 1$ we get two distinct roots $\varphi_+ = \left(1 + \sqrt{5}\right)/2$, $\varphi_- = \left(1 - \sqrt{5}\right)/2$. Having the initial conditions in mind, we know that the coefficients from our general solution must satisfy

$$c_1\varphi_+ + c_2\varphi_- = 1 c_1\varphi_+^2 + c_2\varphi_-^2 = 1 \,.$$

Solving this system of equations for $c_{1,2}$, we obtain $c_1 = 1/\sqrt{5}$, $c_2 = -1/\sqrt{5}$, so we can write the solution as

$$F_n = \frac{\varphi_+ - \varphi_-}{\sqrt{5}} \,.$$

At this point we need to evaluate

$$R \sum_{i=1}^{\infty} \frac{1}{F_n} \,.$$

Resorting to numerical analysis and exploiting fast convergence of this series we get the rounded result as $3.36\,\Omega$.

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Problem M.1 ... stilettos

Which of the following bodies exerts a greater pressure on the ground and what is its value? A cube made of steel, its side 3 m *long or a woman weighing* $59.\bar{9}$ kg *wearing high heels with a diameter of* 5 mm*? Consider the situation when the entire weight of the woman rests on one heel.* *Monika stepped on a bug.*

Let us calculate the pressure exerted by the woman first. We have

$$p_1 = \frac{F_1}{S_1} = \frac{4m_1g}{\pi d^2} \doteq 3.0 \cdot 10^7\,\mathrm{Pa}\,,$$

where we used $g = 9.81\,\mathrm{m{\cdot}s^{-2}}$. In order to calculate the pressure exerted by the cube, we need to know the density of steel ϱ_2. However, there are many varieties of steel, each with a different density. But note that for the cube to exert a pressure larger than p_1 we would need

$$\varrho_2 > \frac{p_1}{ag}\,,$$

where a is the side of the cube. Numerically we have $\varrho_2 > 1.0 \cdot 10^6\,\mathrm{kg{\cdot}m^{-3}}$, which differs from the density of ordinary metals by 3 orders of magnitude. Hence the pressure exerted by the woman must be larger.

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Problem M.2 ... traffic jam

Consider a car moving slowly in a traffic jam with a velocity of $v = 2\,\mathrm{m{\cdot}s^{-1}}$*. Find the rate of change of an angle under which a road sign placed* 2.5 m *above the ground is seen from the car at an initial distance of* $x = 30$ m *from the sign. The dimensions of the car and the road sign are not to be taken into account.* *Verča was observing the behaviour of drivers in a traffic jam.*

With respect to an observer in the car, the road sign moves with a relative horizontal velocity v. The observer sees the sign under the angle φ for which we can write

$$\sin\varphi = \frac{h}{\sqrt{h^2 + x^2}}\,.$$

Hence the component of the velocity of the road sign relative to the car, perpendicular to the line of sight of an observer in the car is

$$v_\mathrm{t} = v\ \sin\varphi\,.$$

Hence the rate of change of φ is

$$\omega = \frac{v_\mathrm{t}}{\sqrt{h^2 + x^2}} = \frac{v\ h}{h^2 + x^2} \doteq 5.5 \cdot 10^{-3}\,\mathrm{s^{-1}}\,.$$

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Problem M.3 ... very slow relativity

Consider two balls moving with velocities $0.4\,\mathrm{m\cdot s^{-1}}$ *and* $0.9\,\mathrm{m\cdot s^{-1}}$ *in the same direction. Before the collision, the slower ball moves ahead of the faster one and after the collision the balls stick together. Find out by how much the temperature of the balls will increase immediately after the collision, assuming that their temperatures before the collision were equal. The specific heat capacity of the material of the balls is* $c = 0.02\,\mathrm{mJ\cdot K^{-1}\cdot g^{-1}}$. *The balls are known to have identical masses.*

Janči was trying to symplify Lukáš's problem on relativity.

From the law of conservation of the momentum, we can get the resulting velocity of the balls as an aritmetic mean of their initial velocities. Then the change in the mechanical energy of the system (which will be entirely converted into heat) is

$$\Delta E = \frac{1}{2}mv_1^2 + \frac{1}{2}mv_2^2 - m\left(\frac{v_1+v_2}{2}\right)^2 = \frac{1}{4}m(v_1-v_2)^2 \,.$$

The change in temperature is equal to the added heat divided by the heat capacity i.e

$$\Delta t = \frac{\Delta E}{2mc} = \frac{(v_1-v_2)^2}{8c} \doteq 1.6\,^\circ\mathrm{C} \,.$$

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Problem M.4 ... splash

Find the minimal height above the water level at which we have to place the bottom end of a vertically oriented rod with length $l = 50\,\mathrm{cm}$ *in order for the entire rod to be submerged under the water when it is dropped. The density of the rod is half the density of water and the rod is slightly weighted at its bottom so that we don't have problems with stability.*

Lukáš was taking a bath.

We denote by ϱ, S, l, x the density of water, area of the cross-section of the rod, its total length and the submerged length respectively. The buoyancy force is then given by

$$F_\mathrm{b} = \varrho S\, xg \,.$$

Work done by this force during the fall of the rod is

$$W = \varrho S\, g \int_0^l x\,\mathrm{d}x = \frac{1}{2}\varrho S\, gl^2 \,.$$

(It is actually the maximal potential energy of a spring with spring constant ϱSg.) Let us choose the level of zero gravitational potential energy just when the rod gets submerged entirely under water. Then it is true that

$$\frac{1}{2}\varrho S\, lgh = \frac{1}{2}\varrho S\, gl^2 \,.$$

The left hand side of the equation represents the gravitational potential energy of the rod at height h relative to the above chosen level of zero gravitation potential energy. But from here it obviously follows that $h = l$, so initially, the bottom end of the rod has to be placed exactly at the water level.

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Problem E.1 ... flash

The battery in Janči's camera has the following specifications written on it: 3.6 V *and* 1 250 mAh. *The capacitor used in the flash has a capacity of* 90 μF *and it is charged to the voltage of* 180 V *each time the flash is used. Assume that during the process of charging the flash, exactly half of the energy is lost. How many pictures with flash can the camera shoot before the battery dies (assuming that initially, the battery was fully charged)? You can assume that the battery is ideal in the sense that its voltage does not decrease during its usage. A fair warning: the result should be an integer.* *Janči hates taking pictures with flash.*

There is an energy of $3.6\,\mathrm{V} \cdot 1.250\,\mathrm{Ah} \cdot 3\,600\,\mathrm{s{\cdot}h^{-1}}$ joules stored in the battery. One usage of flash requires (using the standard notation) the energy CU^2. The ratio between the energy needed for one flash and the total amount of energy available in the battery is 5 555 (rounded down). Hence we run out of the energy after this number of flashes.

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Problem E.2 ... tangled resistances

Find the current I through the source with voltage $U = 1\,\mathrm{V}$ assuming that each resistor has resistance $1\,\Omega$. *Janči was trying to think of a task, but not about the solution.*

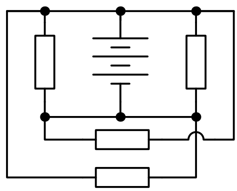

Fig. 6: Schematics

The lower two junctions where the three wires meet can be connected by another wire so that the meshed wires get disentangled. Redrawing the circuit, we see, that the source is connected in parallel to a pair of pairs resistors connected again in parallel. Hence the effective resistance of the circuit is one quarter of the resistance of one of the resistors. The current through the source is then 4 A.

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Problem E.3 ... aberration

Light with a wavelength of 400 nm *in vacuum has a wavelength of* 265 nm *when it passes through a lens, while light with a wavelength of* 700 nm *in vacuum has a wavelength of* 460 nm *when it passes through the same lens. Using blue light (vacuum wavelength* 400 nm*), the focal length of the lens is* 1 m. *By how much will the focal length change if we use red light (vacuum wavelength* 700 nm*) instead? Assume that the lens is thin.*

Janči trying to remember the parts of optics he actually liked.

Let us start with the lensmaker's equation for a thin lens

$$\frac{1}{f} = \frac{n - n_0}{n} \left(\frac{1}{R_1} - \frac{1}{R_2} \right) ,$$

where n_0 is the refractive index of the lens' surroundings (in the case of vacuum we have $n_0 = 1$) and n is the refractive index of the lens itself. We can write down two instances of this equation, one for each wavelength. Denoting the ensuing focal lengths by f_r (red light) and f_b (blue light) and dividing these two lensmaker's equations through one another, we obtain

$$f_\mathrm{r} = \frac{n_\mathrm{b} - 1}{n_\mathrm{r} - 1} f_\mathrm{b} ,$$

where the corresponding refractive indices n_r and n_b can be determined using $\lambda = \lambda_0 / n$ where λ_0 is the vacuum wavelength and λ is the wavelength inside the lens. Hence the change in focal length is

$$f_\mathrm{r} - f_\mathrm{b} = f_\mathrm{b} \left(\frac{\lambda_\mathrm{r} \left(\lambda_\mathrm{b0} - \lambda_\mathrm{b} \right)}{\lambda_\mathrm{b} \left(\lambda_\mathrm{r0} - \lambda_\mathrm{r} \right)} - 1 \right) \doteq 0.024\,\mathrm{m} .$$

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Problem E.4 ... useless work

Consider a parallel-plate capacitor charged to a voltage of $1\,\mathrm{V}$*. In a direction perpendicular to the electric field lines between the plates, there is a magnetic field of* $2\,\mathrm{\mu T}$*. Distance between the two plates is* $d = 0.1\,\mathrm{mm}$*. Assume that an electron is emitted from the plate at lower potential with zero initial speed. Find the speed at which it arrives at the second plate.*

Xellos likes to be in the way.

Since magnetic field does not do any work the electron will gain kinetic energy $E = 1\,\mathrm{eV}$. In terms of speeds this means that

$$v = \sqrt{\frac{2E}{m_\mathrm{e}}} \doteq 593\,000\,\mathrm{m \cdot s^{-1}} .$$

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Problem X.1 ... let's spread out

At some temperature T, the cubic lattice of an iron crystal undergoes a transition from body-centred to face-centred phase. During this transition, the length of the unit cell edge increases by 22 %. Find the factor by which the density of the crystal will decrease.

Tomáš Bárta was thinking about iron.

In the body-centred phase, two atoms correspond to each cell while in the face-cetred phase, four atoms correspond to each. The mass corresponding to one cell will therefore double in magnitude and its volume will increase by the factor of 1.22^3. Hence the density of the crystal will decrease by the factor of $2/1.22 \cdot 10^3 \doteq 1.10$.

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Problem X.2 ... heating season

How many times do we have to let a bouncy ball with a mass of $m = 200\,\mathrm{g}$ fall on the ground (made of polyvinyl chloride with a density of $\varrho = 1380\,\mathrm{kg{\cdot}m^{-3}}$ and a specific heat capacity $c = 0.9\,\mathrm{kJ{\cdot}kg^{-1}{\cdot}K^{-1}}$) from the height $h = 1\,\mathrm{m}$ so that a thermally well-isolated piece of the ground with area $S = 1\,\mathrm{m^2}$ and thickness $d = 1\,\mathrm{cm}$ would increase its temperature by one degree of Celsius? The coefficient of restitution k for collisions between the bouncy ball and the ground, defined as the ratio of the speed of the bouncy ball after a collision and its speed before the collision, is known to be 70 %. Use the following value for the acceleration due to gravity: $g = 9.81\,\mathrm{m{\cdot}s^{-2}}$.

Terka trying to warm her frozen feet by alternative means.

Before a collision with the ground, the bouncy ball has a total mechanical energy of mgh, which deacreases to $mghk^2$ after the collision. The amount of heat transferred to the ground during one collision is therefore equal to $mgh(1-k^2)$. If we denote by N the total number of collisions needed, we can write

$$Nmgh(1-k^2) = c\varrho S\, d\Delta T\,,$$

$$N = \frac{c\varrho S\, d\Delta T}{mgh(1-k^2)} \doteq 12\,413\,,$$

where we rounded the numerical value of N up to the nearest integer.

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Problem X.3 ... hexagonal sphere

The lattice of an alpha-boron nitride crystal consists of atomic layers in which the atoms are arranged in a hexagonal structure. We managed to acquire a sphere with radius $r = 1\,\mathrm{\mu m}$ made of this material. Find the maximum number of layers which can be intersected by a straight line intersecting the sphere. You are encouraged to look up any information needed.

Xellos having nightmares about chemistry.

The distance between individual layers can be found as $c = 6.66\,\text{Å}$ (e.g. `http://www.ioffe.rssi.ru/SVA/NSM/Semicond/BN/basic.html`). Clearly, the maximum number of layers will be intersected by a diameter of the sphere. Hence we find the answer to be

$$N = \frac{2r}{c} \doteq 3\,000\,.$$

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Problem X.4 ... firm or springy

Pistons in the fork of a bicycle have a cross-section of $S = 5\,\mathrm{cm}^2$ and height $h = 20\,\mathrm{cm}$. Assume that a manometer connected to the fork gives a reading of $p_1 = 100\,\mathrm{psi}$. By how much will the fork be compressed (in cm), when a $m = 60\,\mathrm{kg}$ man leans on it with all his weight? Assume ideal diatomic gas behaviour and that an adiabatic process takes place. Do not forget that the bike fork has two arms. *Mirek's been avoiding fixing his bike for some time already.*

First, let us write down the equation for adiabatic process with an ideal gas

$$p_1 V_1^\kappa = p_2 V_2^\kappa\,,$$

where κ is the Poisson's constant. Initially, the volume was $V_1 = Sh$ and after the compression, it changed to $V_2 = S(h - \Delta h)$, where Δh is the compression of the fork we are looking for. For the corresponding pressures we can write

$$p_2 = p_1 + \frac{mg}{2S}\,,$$

where the factor of a half comes from distributing the force into both arms of the fork. Substituting the relations for p_2, V_1, V_2 and calculating $1/\kappa$ power, we get

$$p_1^{1/\kappa} Sh = \left(p_1 + \frac{mg}{2S}\right)^{1/\kappa} S(h - \Delta h)$$

and rearranging

$$\Delta h = h\left(1 - \left(\frac{p_1}{p_1 + \frac{mg}{2S}}\right)^{1/\kappa}\right)\,.$$

Substituting $\kappa = 7/5$ and $1\,\mathrm{psi} = 6\,895\,\mathrm{Pa}$ yields $\Delta h \doteq 7.1\,\mathrm{cm}$.

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