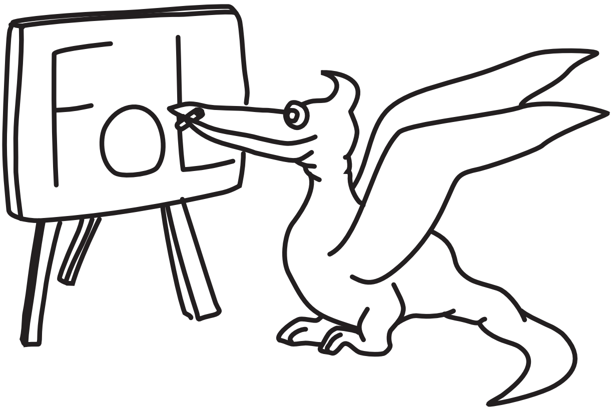
Solution of 4th Online Physics Brawl



Problem FoL.1 ... thoughtful

While sitting and thinking deeply, Aleš was throwing a little ball against a wall. Each time, he threw the ball with an initial speed $v_0 = 10\,\mathrm{m{\cdot}s^{-1}}$ at an angle $\alpha = 60^\circ$ from the vertical axis. Since he was sitting in a narrow corridor with width $d = 1.5\,\mathrm{m}$, the ball always bounced back and forth between the walls a couple of times. What is the maximum height (measured relative to ball's initial position) that the ball will reach along its trajectory? Consider all collisions to be perfectly elastic. The acceleration due to gravity is $g = 9.81\,\mathrm{m{\cdot}s^{-2}}$.

Mirek's mechanical mental image of periodic boundary conditions.

The ball leaves Aleš's hand at an elevation angle of $\pi/2-\alpha$. Initially, the ball follows a parabolic trajectory. When the ball hits the wall, the vertical component of its momentum is conserved and the horizontal one just flips its sign. Therefore, the ball will still follow a parabolic trajectory, which, however, will be mirrored at the point where the ball hits the wall. Therefore, the maximum height can be computed the same way as usually in the case without walls.

Yet, there is an easier path to follow, which makes use of conservation of energy. Assuming that the horizontal component of the ball's momentum is conserved, we can transform the problem into that of a projectile fired vertically upwards with initial speed $v_0 \sin(\pi/2 - \alpha)$. The maximum height reached by the ball is therefore

$$h_{\max} = v_0^2 \sin^2(\pi/2 - \alpha)/(2g) \doteq 1.27\,\mathrm{m}\,.$$

Hence, the ball climbs to the height $h_{\max} \doteq 1.27\,\mathrm{m}$ above its initial position.

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Problem FoL.2 ... Ded Moroz

Consider Mr Spherical (whose shape, incidentally, happens to be perfectly spherical) that likes to wear a tight-fitting jacket of negligible thickness (compared to his radius). Wearing his jacket, Mr Spherical went outside for a walk. Once the thermal equilibrium between Mr Spherical and his surroundings was established, the temperatures at the inner and outer surfaces of his jacket could be measured to be $25\,^\circ\mathrm{C}$ and $-5\,^\circ\mathrm{C}$, respectively. Find the temperature (in degrees Celsius), which is measured exactly in the middle of the layer of Mr Spherical's jacket (i.e. half-way through the bulk of the layer).

Sometimes, Lubošek feels really cold.

Since the thickness of the layer is negligible compared to Mr Spherical's radius, the boundary surfaces may be well approximated by planes, on which the temperature is kept constant. In such a case, the steady solution of the heat equation is linear in position inside the layer, so the temperature half-way through must be simply the arithmetic mean of the temperatures at both boundaries, i.e. $10\,^\circ\mathrm{C}$.

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Problem FoL.3 ... fiat nox

Consider a short-circuited two-conductor copper wire. At one end of the wire, a resistance of $R = 65.3\,\mathrm{m\Omega}$ was measured using an ohmmeter. What is the distance (along the wire) to the short-circuited section of the wire from the end where the measurement was performed?

The resistivity of copper is $\varrho = 1.71 \cdot 10^{-8}\,\Omega{\cdot}\mathrm{m}$, *the cross-sectional diameter of one conductor is* $d = 1.50\,\mathrm{mm}$.

A switch was tripped in Mirek's room.

The total resistance R of a conductor with length l, cross-sectional area $S = \pi d^2/4$ and resistivity ϱ is given by

$$R = \frac{\varrho l}{S}\,.$$

We must not forget that in our case, we are dealing with a two-conductor wire, so the distance along the wire is $l/2$. Hence, the distance between the short-circuited section and the place where the resistance was measured is

$$\frac{l}{2} = \frac{RS}{2\varrho} = \frac{R\pi d^2}{8\varrho} = 3.37\,\mathrm{m}\,.$$

Wanted distance is equal to 3.37 m.

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Problem FoL.4 ... the runnier, the better

Dominika is secretly preparing homeopathics at home. She uses a cup filled with a solution of volume $V_0 = 200\,\mathrm{ml}$ *and concentration of the active agent* $c_0 = 0.001\,\mathrm{M}$. *She then pours away a half of this volume and fills the cup with water to the original volume* V_0. *She repeats this process 70 times. Statistically speaking, how many molecules of the active agent are left in the cup at the end? Assume that the solution and water are always perfectly mixed.*

Mirek heard some gossip.

Assuming that the water and the solution are always perfectly mixed, the amount of the active agent in the solution is halved each time we repeat the process. Therefore, after i repetitions, the concentration of the solution is

$$c_i = \frac{c_0}{2^i}\,.$$

The amount of substance, given the known concentration and volume, can be written as

$$n_i = c_i V_0\,.$$

The number of molecules N_i is then given by multiplying this by the Avogadro constant N_A, i.e.

$$N_i = N_\mathrm{A} c_i V_0 = 2^{-i} N_\mathrm{A} c_0 V_0\,.$$

Finally, for $i = 70$ we get

$$N_{70} \doteq 0.10\,.$$

Hence, statistically speaking, only one tenth of a molecule of active agent is left in the cup.

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Problem FoL.5 ... brake failure

A truck experiences brake failure and needs to make an emergency stop. Luckily enough, it's just driving through a flood zone with lots of sand-filled bags. Assume that the mass of the truck is $M = 6\,\mathrm{t}$ and that one bag filled with sand weighs $m = 60\,\mathrm{kg}$. How many sand bags will the truck need to collide with until it slows down to $75\,\%$ of its initial speed $v_0 = 50\,\mathrm{km{\cdot}h^{-1}}$? Assume that the collisions are perfectly inelastic and that after colliding with the truck, the bags are pushed in front of it without friction.

Mirek was forced to replace people with sand bags.

Let n denote the number of collisions needed to slow the truck down. Also, denote the speed of the truck after n collisions by v_n and set $\alpha = v_n/v_0$. Conserving linear momentum, we can write

$$Mv_0 = (M+m)v_1 = (M+2m)v_2 = \ldots = (M+nm)v_n\,.$$

From this expression, we get

$$\frac{v_n}{v_0} = \alpha = \frac{M}{M+nm}\,,$$

where[1]

$$n = \left\lceil \frac{M}{m}\left(\frac{1}{\alpha}-1\right)\right\rceil = 34\,.$$

Hence, the truck will slow down to $75\,\%$ of its initial speed after colliding with 34 sand bags.

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Problem FoL.6 ... fungus librariensis

In order to protect books from becoming damp, some libraries choose to keep the relative humidity at $\varphi_0 = 25\,\%$. Find the mass of water which would (after complete evaporation) turn such a library into a suitable environment for mould growth. You can assume that the ideal value of relative humidity for mould growth is $\varphi_p = 60\,\%$. The total volume of air in this particular library is $V = 180\,\mathrm{m^3}$, the (constant) ambient temperature is $t = 24\,^\circ\mathrm{C}$ and the saturated vapour density at $24\,^\circ\mathrm{C}$ is $21.8\,\mathrm{g{\cdot}m^{-3}}$.

Lydka likes to daydream about her private library.

Let us start with the defining relation for relative humidity,

$$\Phi = \varphi\Phi_{\max}\,,$$

where Φ is the absolute humidity and $\Phi_{\max}$ is the mass of saturated vapour in a given volume at a given temperature. Also,

$$\Phi = \frac{m}{V}\,,$$

where m is the total mass of water vapour in volume V, so

$$m = \varphi\Phi_{\max}V\,.$$

[1] The notation $\lceil x\rceil = \min\{m \in \mathbb{Z} | m \geq x\}$ is used for the ceiling function.

Denoting by m the mass of water required to increase the relative humidity from φ_0 to φ_p, we have $m = m_\mathrm{p} - m_0$, where m_0 and m_p are the total masses of water vapour at relative humidities φ_0 and φ_p, respectively. After substituting, we obtain

$$m = \Phi_{\max} V(\varphi_\mathrm{p} - \varphi_0) \doteq 1.37\,\mathrm{kg}\,.$$

Hence, in order to turn the given library into an ideal place for mould growth, we would need to evaporate 1.37 kg of water.

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Problem FoL.7 ... server not found

Consider computer X, which is connected by two optical fibres to two other computers A and B. The total length of these optical fibres together is $l = 3.0\,\mathrm{km}$. A signal from computer A, bearing a message that it was sent at the time $1\,413\,753\,899.213\,575\,3\,\mathrm{s}$ (Unix timestamp of the time counter at computer A) reached computer X at the time $1\,413\,753\,899.213\,592\,1\,\mathrm{s}$ (Unix timestamp of the time counter at computer X). Similarly, a signal from computer B was received by computer X at time $1\,413\,753\,906.459\,900\,4\,\mathrm{s}$, bearing the message that it was sent at the time $1\,413\,753\,906.459\,892\,5\,\mathrm{s}$. Assume that the time counters of computers A and B are synchronized, but that the time counter of computer X is either gaining or lagging behind. Luckily, this deviation stays constant in time. Also, do not take into account either the effects of relativity or the time to process the signals. Assuming that the speed of propagation of signals along the fibres is $c = 2.0 \cdot 10^8\,\mathrm{m{\cdot}s^{-1}}$, find the length of the fibre between A and X.

Michal was trying to reduce the dimensions of GPS equations.

Let Δ be the deviation of the local time counter from the synchronized counters of computers A and B. Also, denote by s_b the time at which the signal was sent from computer B (measured by its time counter) and by r_b the time at which the signal was received from computer B (measured by the counter of the receiving computer). The distance x then satisfies

$$cr_a = cs_a + x + c\Delta\,,$$
$$cr_b = cs_b + l - x + c\Delta\,.$$

Solving for x, we obtain

$$x = \frac{c(r_a - s_a - (r_b - s_b)) + l}{2}\,,$$

so, plugging in the numbers, we get $x \doteq 2\,390\,\mathrm{m}$.

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Problem FoL.8 ... Archimedes drowned

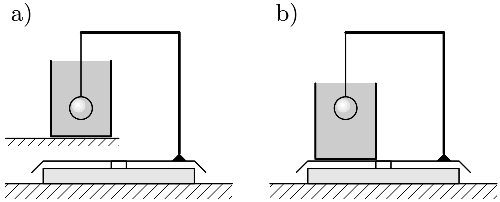
The diagram below shows schematically two situations dealing with an aluminium sphere, cylindrical flask with water and a holding apparatus from which the sphere can be suspended. In the two situations, we have that

a) *the holder is placed on top of the scales with the sphere suspended from it. The sphere is fully submerged in the flask which is held off the scales,*

b) the same as a), except in this case, the flask rests on the scales.
Find the difference $\Delta m = m_b - m_a$ of the readings m_b and m_a shown by the scales in the situation b) and a) respectively.

The radius of the (homogeneous) sphere is $r = 1.03\,\mathrm{cm}$, the inner radius of the flask is $R = 3.01\,\mathrm{cm}$. The height of the water column in the flask with the sphere fully submerged in it is $h = 5.10\,\mathrm{cm}$. The densities of water and aluminium are $\varrho = 996\,\mathrm{kg{\cdot}m^{-3}}$ and $\varrho_{Al} = 2\,700\,\mathrm{kg{\cdot}m^{-3}}$, respectively. The mass of an empty flask is $m_n = 124.5\,\mathrm{g}$. Assume that the suspension part of the holding apparatus is of negligible mass and volume. The rest of the apparatus has a mass of $m_d = 523.5\,\mathrm{g}$. The range of values that the scales can measure is sufficient for it to show both readings with accuracy up to $0.1\,\mathrm{g}$. The density of air should be regarded as negligible. State your answer in kilograms.

Karel wanted to set a simple-minded but computationally horrendous problem.



Let us first realise the effects to be considered in both situations. In the situation

a) there is a holder placed on top of the scales from which the sphere is suspended. However, since the sphere is submerged in water, buoyancy produces a small lift, which partially balances gravity. Hence, the reading in this case is

$$m_a = m_\mathrm{d} + (\varrho_\mathrm{Al} - \varrho)\,V_\mathrm{k}\,,$$

where $V_\mathrm{k} = 4\pi r^3/3$ is the volume of the sphere.

b) we have the holder, sphere and flask all resting on top of the scales, so the reading is

$$m_b = m_\mathrm{d} + \varrho_\mathrm{Al} V_\mathrm{k} + m_\mathrm{n} + \varrho V_\mathrm{v}\,,$$

where $V_\mathrm{v} = \pi h R^2 - 4\pi r^3/3$ is the volume of water in the flask.
It'd be more useful to express the volume of water in the flask in terms of the volume V_n of a cylinder with height h and radius R. Then we have $V_\mathrm{v} = V_\mathrm{n} - V_\mathrm{k}$, so

$$m_b = m_\mathrm{d} + \varrho_\mathrm{Al} V_\mathrm{k} + m_\mathrm{n} + \varrho V_\mathrm{n} - \varrho V_\mathrm{k}\,.$$

The required difference $\Delta m = m_b - m_a$ is then easily found to be

$$\Delta m = m_b - m_a = m_\mathrm{n} + \varrho V_\mathrm{n} = m_\mathrm{n} + \pi \varrho h R^2 \doteq 0.269\,1\,\mathrm{kg} = 269.1\,\mathrm{g}\,.$$

The difference of the readings shown by the scales in the two situations expressed in basic SI units is therefore $0.269\,1\,\mathrm{kg}$.

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Problem FoL.9 ... carriage on a blind track

Consider a railway carriage with mass $M = 20\,\mathrm{t}$, which moves on the tracks without friction. Inside the carriage, there is an elephant with mass $m = 4\,\mathrm{t}$, initially at rest. Find the distance by which the carriage moves, if the elephant walks a distance of $l = 12\,\mathrm{m}$ and then stops again. (The elephant can move only parallel to the tracks, because the carriage is quite narrow.)

Michal thinks that elephants are cool.

The key fact to realise is that the centre of mass of the combined system consisting of the carriage and the elephant does not move, since there are no external forces acting. The elephant's mass is $m/(m+M) = 1/6$ of the whole system's mass. When the elephant walks a distance l, the center of mass moves in the carriage's reference frame by $l/6 = 2\,\mathrm{m}$. Hence, for an external observer, the carriage moved by $2\,\mathrm{m}$ in the direction opposite to that of the elephant's movement.

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Problem FoL.10 ... driving in rain

Find the total volume of water (in litres) that falls down on a Karosa 700 *type bus as it travels for an hour at a speed of $v_\mathrm{b} = 50\,\mathrm{km{\cdot}h^{-1}}$. Assume the bus to be of a cuboid shape with height $h = 4\,\mathrm{m}$, width $w = 2.5\,\mathrm{m}$ and length $l = 12\,\mathrm{m}$. The rain falls vertically downwards with speed $v_\mathrm{r} = 10\,\mathrm{m{\cdot}s^{-1}}$ and flux (that is, the volume of water crossing a unit area perpendicular to the velocity of the falling rain, per unit time) $Q = 5\,\mathrm{mm{\cdot}h^{-1}}$.* *Xellos running from the the rain.*

Let us adopt the point of view of an observer, which is fixed in the reference frame connected with the falling droplets of rain. In this frame, the droplets are fixed and the bus gains an upward component of its velocity, equal to v_d. It is therefore an equivalent problem to consider the spatial volume swept out by the leading and the upper face of the bus over $t = 1\,\mathrm{h}$ in this frame and calculate the volume of water in this spatial volume.

To this end, considering (in the original frame connected with the ground) a cube with side-length a (through which the droplets pass in a time of a/v_d), which contains a total volume

$$Qa^2 \frac{a}{v_\mathrm{d}} = a^3 \frac{Q}{v_\mathrm{d}}$$

of water, we can deduce that in the frame, where the droplets are static, a spatial volume V contains a total volume VQ/v_d of water.

Clearly, the upper face of the bus sweeps out a rectangular parallelepiped with base surface area wl and height $v_\mathrm{d}t$, hence with volume $V_1 = wlv_\mathrm{d}t$. The leading face of the bus sweeps out another rectangular parallelepiped, this time with base surface area wh and height $v_\mathrm{a}t$, hence with volume $V_2 = whv_\mathrm{a}t$. Putting all results together, the volume of water that falls down on the bus in time t is

$$V_\mathrm{d} = (V_1 + V_2)\frac{Q}{v_\mathrm{d}} = Qw\left(l + \frac{v_\mathrm{a}}{v_\mathrm{d}}h\right)t \doteq 219\,\mathrm{l}\,.$$

Hence, as the bus travels for an hour, a volume of $219\,\mathrm{l}$ of water falls down on it.

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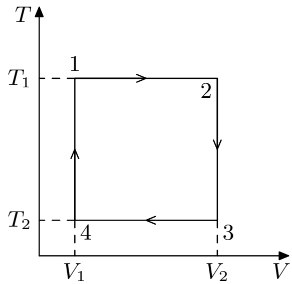
Problem FoL.11 ... on your own drive

Find the efficiency (in %) of a thermodynamic cycle consisting of two isothermal processes and two isochoric processes (see the figure). Assume that the cycle works between temperatures $T_1 = 373\,\mathrm{K}$ *and* $T_2 = 273\,\mathrm{K}$ *and that the heat released during the isochoric process* $2 \to 3$ *is fed into the isochoric process* $4 \to 1$. *Xellos spied on a hamster rolling his wheel.*



No work is done during an isochoric prosess, therefore the heat released by cooling the system from temperature T_2 to T_1 is the same as the heat needed to do the opposite process of heating the system from temperature T_1 to T_2. Hence, the contributions due to the two isochoric processes exactly cancel out and do not enter the calculation of efficiency. The work done by the system during the expansion process $1 \to 2$ (during which the volume increases from V_1 to V_2) is

$$W_1 = nRT_1 \ln \frac{V_2}{V_1}\,;$$

an equal amount of heat $Q = W_1$ must be supplied to the system. Also, the work done during the contraction process is $3 \to 4$ is $W_1 = -nRT_2 \ln(V_2/V_1)$ where this time, no additional heat is required. The efficiency can then be found as

$$\eta = \frac{W_1 + W_2}{Q_1} = \frac{T_1 - T_2}{T_1} \doteq 0.268 = 26.8\,\%\,.$$

The efficiency of our thermodynamic cycle is therefore equal to that of Carnot's cycle working between temperatures T_1 and T_2.

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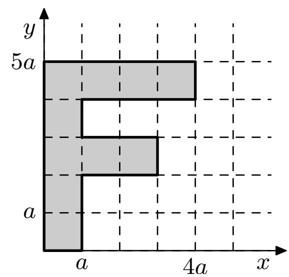
Problem FoL.12 ... the most important letter's centre of mass

Has it ever occurred to you that F *might actually be the most important letter of the alphabet? Find the* y *coordinate of the centre of mass of the shape of the letter* F. *You can assume that the shape is built up from 10 identical homogeneous square-shaped laminae with side length* a. *Express your answer to three significant figures as a multiple of* a *in the coordinate grid indicated in the figure.* *Karel contemplated on the importance of the letter* F.



The y coordinate of the centre of mass satisfies

$$m_{\mathrm{tot}} y = \sum m_i y_i\,,$$

where m_{tot} is the total mass of the shape. Taking into account homogeneity of each constitutive lamina, we can assert that the centre of mass of each of them lies at its geometrical centre (that is, at a distance of $0.5a$ from their sides). Denoting by $m = m_{\mathrm{tot}}/10$ the mass of each lamina, the above expression for y can be written as

$$10my = ma(1 \cdot 0.5 + 1 \cdot 1.5 + 3 \cdot 2.5 + 1 \cdot 3.5 + 4 \cdot 4.5)\,,$$
$$y = 3.1a\,.$$

Hence, the correct answer expressed to 3 significant figures is $y = 3.10a$. An analogous calculation yields $x = 1.4a$ for the x coordinate.

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Problem FoL.13 ... sulphuring the barrels

Sulphur dioxide was and still is used as a preservative for wine casks. One of the sulphuring techniques involves burning sulphur discs directly inside the casks. Consider a cask with volume $200\,\mathrm{l}$, *inside which two sulphur discs are placed, each containing* $2\,\mathrm{g}$ *of sulphur. The discs are then ignited and the cask is closed hermetically. What is the fraction of cask's volume that* SO_2 *will occupy after both discs burn down completely? Assume that when all burning stops, the temperature inside the cask decreases immediately back to* $T = 20\,^\circ\mathrm{C}$, *the same as it was initially. Molar mass of sulphur is* $M = 32\,\mathrm{g{\cdot}mol^{-1}}$ *and the ambient atmospheric pressure is* $p = 101\,\mathrm{kPa}$. *Mirek knows a thing or two about physicists' views on chemical calculations.*

Sulphur burning is described by the chemical equation

$$\mathrm{S} + \mathrm{O_2} \longrightarrow \mathrm{SO_2}\,.$$

Therefore, one mole of sulphur becomes exactly one mole of sulphur dioxide. The volume occupied by one mole of gas under the given conditions is

$$V_\mathrm{m} = \frac{RT}{p}\,.$$

Also, the amount of SO_2 can be written as

$$n_{\mathrm{SO_2}} = \frac{2m}{M}\,,$$

where m denotes the mass of sulphur in one disc. The fraction of cask's volume we are trying to express is simply the ratio of $V_{\mathrm{SO_2}}$ to the total volume V of the cask. Hence

$$\varphi = \frac{V_{\mathrm{SO_2}}}{V} = \frac{n_{\mathrm{SO_2}} V_\mathrm{m}}{V} = \frac{n_{\mathrm{SO_2}} RT}{pV} \doteq 0.015\,1$$

The wanted fraction of SO_2 in cask is 0.015 1.

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Problem FoL.14 ... to jump, or not to jump

A horizontal plate performs simple harmonic motion with period $T = 0.5\,\mathrm{s}$ *in the vertical direction. Find the maximum value of the amplitude of oscillations* A *(in centimetres) such that a body lying on the plate would remain in contact with the plate throughout the motion. The acceleration due to gravity is* $g = 9.81\,\mathrm{m{\cdot}s^{-2}}$.

f(Aleš) salvaged this from an ancient book.

The body lying on the plate will clearly stay in contact with the plate over the course of the motion, provided that the magnitude of acceleration of the plate at the upper turning point of its motion remains less than the magnitude of acceleration due to gravity. The vertical position y of the oscillating plate is given by

$$y = A\sin\omega t\,,$$

where we take the y axis to point upwards. The acceleration a of the plate at a given time t can be easily determined by differentiating the above expression for y twice with respect to time, i.e.

$$a = -\omega^2 A\sin\omega t\,.$$

We have $t = T/4$ at the upper turning point, so the condition for the body to remain in contact with the plate reads

$$-g = -\omega^2 A\sin\omega\frac{T}{4}\,.$$

Finally, substituting $\omega = 2\pi/T$ and $\sin\pi/2 = 1$ we have

$$A = \frac{gT^2}{4\pi^2} \doteq 6.2\,\mathrm{cm}\,.$$

Hence, the body remains in contact with the plate if the amplitude of oscillations does not exceed $6.2\,\mathrm{cm}$.

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Problem FoL.15 ... tea at five

Náry decided to make himself a cup of mint tea. Reading the instructions „Place the teabag into boiling water.“ on the side of the box made him think about the following question: How much energy could be saved, if only he could prepare his tea at the summit of Triglav, the highest peak in Slovenia (2 864 meters above the sea level) instead of doing it at his home in Boskovice? You can assume that the ambient pressure and temperature in Boskovice are $p_{\mathrm{B}} = 101\,325\,\mathrm{Pa}$ and $t_{\mathrm{B}} = 25\,^{\circ}\mathrm{C}$ respectively. The temperature at Triglav is $t_T = 13\,^{\circ}\mathrm{C}$ and ideal gases there occupy 1.44 times the volume they occupy in Boskovice. Also, the boiling point of water t_{v} and the ambient pressure p are related (over a suitable range of values of p) as

$$t_{\mathrm{v}} = \tau + kp\,,$$

where $\tau = 71.6\,^{\circ}\mathrm{C}$ and $k = 2.8\cdot 10^{-4}\,\mathrm{K{\cdot}Pa^{-1}}$. In order to prepare his tea, Náry needs to bring $m = 600\,\mathrm{g}$ of water with initial temperature $t_0 = 20\,^{\circ}\mathrm{C}$ (assumed to be the same in both cases) to a boil. All energy losses should be neglected. The specific heat capacity of water is $c = 4\,180\,\mathrm{J{\cdot}kg^{-1}{\cdot}K^{-1}}$. *Kiki and Tom pondering about their excessive consumption of tea.*

So as to find the boiling point of water at Triglav, let us first determine the ambient pressure p_{T} at Triglav. Using the Ideal Gas Law, we can write

$$\frac{p_{\mathrm{B}}V_{\mathrm{B}}}{T_{\mathrm{B}}} = \frac{p_{\mathrm{T}}V_{\mathrm{T}}}{T_{\mathrm{T}}}\,,$$

where the quantities denoted by subscripts B and T correspond to the values of pressure, volume and temperature in Boskovice and at Triglav, respectively. The pressure at Triglav may then be expressed as

$$p_{\mathrm{T}} = p_{\mathrm{B}}\frac{V_{\mathrm{B}}}{V_{\mathrm{T}}}\frac{T_{\mathrm{T}}}{T_{\mathrm{B}}} = \frac{p_{\mathrm{B}}T_{\mathrm{T}}}{pT_{\mathrm{B}}}\,.$$

The difference Δt between the boiling points t_{vB} and t_{vT} in Boskovice and at Triglav, respectively, is then proportional to the amount ΔQ of energy saved, since we are assuming that, in both situations, we boil water with the same initial temperature (irrespective of the fact that the ambient temperatures at the two sites are different) and we neglect all losses of energy. The answer can then be found as

$$\Delta Q = mc\Delta t\,,$$

where

$$\Delta t = t_{\mathrm{vB}} - t_{\mathrm{vT}} = k(p_{\mathrm{B}} - p_{\mathrm{T}}) = kp_{\mathrm{B}}\left(1 - \frac{T_{\mathrm{T}}}{pT_{\mathrm{B}}}\right)$$

and so

$$\Delta Q = mckp_{\mathrm{B}}\left(1 - \frac{T_{\mathrm{T}}}{pT_{\mathrm{B}}}\right)\,.$$

Numerically, we get $\Delta Q \doteq 23\,700\,\mathrm{J}$.

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Problem FoL.16 ... scary shopping

Kiki set off to London (which lies $s = 1\,500\,\mathrm{km}$ from her home) to buy some $^{18}\mathrm{F}$. Knowing that the half-life of this poor quality stuff was only $1.8\,\mathrm{h}$, she decided to buy $32\,\mathrm{g}$ right away. Find the speed (in $\mathrm{km{\cdot}h^{-1}}$) with which she had to fly back in order to bring home $10\,\mathrm{g}$ of $^{18}\mathrm{F}$ more than if she had opted for a 24 hour journey by bus. *Dominika had to go and claim her money back already a third time this year, wondering why anyone else's shoes last so much longer.*

Half-life is defined to be the time needed for a half of the total number of particles in a given sample to decay. The total mass m of $^{18}\mathrm{F}$ atoms in a sample after a time t (directly proportional to the total number N of atoms which have not decayed after time t) then satisfies

$$m = m_0 2^{-t/T}\,,$$

where m_0 is the total mass of $^{18}\mathrm{F}$ atoms in the sample at $t = 0$ and T is the half-life of $^{18}\mathrm{F}$. The difference $\Delta m = 10\,\mathrm{g}$ in the total mass of $^{18}\mathrm{F}$ atoms brought home (which is achieved by Kiki travelling faster) can be expressed as

$$\Delta m = m_0\left(2^{-t_{\mathrm{l}}/T} - 2^{-t_{\mathrm{b}}/T}\right)\,,$$

where t_{l} and t_{b} are the times it takes the plane and the bus respectively to travel $s = 1500\,\mathrm{km}$. Clearly $t_{\mathrm{l}} = s/v_{\mathrm{l}}$, where v_{l} is the quantity we are interested in. Substituting for $t_{\mathrm{l}} = s/v_{\mathrm{l}}$ into the above expression for Δm we get

$$v_{\mathrm{l}} = \frac{s}{T\log_2\dfrac{1}{\dfrac{\Delta m}{m_0} + 2^{-t_{\mathrm{b}}/T}}}\,.$$

This can be simplified by realising that $2^{-t_{\mathrm{b}}/T} \ll \Delta m/m_0$. Then we can write

$$v_{\mathrm{l}} \approx \frac{s}{T\log_2\dfrac{m_0}{\Delta m}} \doteq 500\,\mathrm{km{\cdot}h^{-1}}\,.$$

Hence, so as to bring home 10 g of ^{18}F more, the speed of the plane would have to be $v_1 = 500\,\mathrm{km\cdot h^{-1}}$.

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Problem FoL.17 ... feed a castle

In their time (i.e. at the turn of the 19th and 20th century), the owners of the Kokořín castle were progressively thinking people – they introduced gas heating based on burning acetylene created by reacting calcium carbide with water

$$\mathrm{CaC_2} + 2\,\mathrm{H_2O} \longrightarrow \mathrm{C_2H_2} + \mathrm{Ca(OH)_2}\,.$$

Compare the energetic efficiency of this carbide-fuelled heating to that of a wood-fuelled alternative, i.e. find the ratio of energies gained from a unit mass of wood and a unit mass of carbide. Use the following data: density of calcium carbide $\varrho(\mathrm{CaC_2}) = 2\,200\,\mathrm{kg\cdot m^{-3}}$, molar mass of calcium carbide $M(\mathrm{CaC_2}) = 64\,\mathrm{g\cdot mol^{-1}}$, heating value of acetylene $H(\mathrm{C_2H_2}) = 48\,\mathrm{MJ\cdot kg^{-1}}$, molar mass of acetylene $M(\mathrm{C_2H_2}) = 26\,\mathrm{g\cdot mol^{-1}}$, heating value of wood $H_d = 14\,\mathrm{MJ\cdot kg^{-1}}$, density of wood $\varrho = 520\,\mathrm{kg\cdot m^{-3}}$. Assume that the castle has a sufficiently large water reservoir. The fact that the fuel needs to be transported over an altitude difference $h = 50\,\mathrm{m}$ from a nearby lying village uphill should be included in your energy balance considerations. The acceleration due to gravity is $g = 9.8\,\mathrm{m\cdot s^{-2}}$. *Mirek and Lukáš on a trip.*

Assume that we have a mass m of each of the two kinds of fuel at our disposal. Then, the net energies gained from each fuel are

$$E_\mathrm{d} = mH_\mathrm{d} - mgh$$
$$E(\mathrm{CaC_2}) = m\frac{M(\mathrm{C_2H_2})}{M(\mathrm{CaC_2})}H(\mathrm{C_2H_2}) - mgh\,,$$

where we already converted the mass of carbide to the corresponding mass of acetylene. Also, the energy needed to transport the fuel over the given altitude difference must be subtracted from the heat gained by burning in order to obtain the net energy gain. By inspecting the given data, it follows immediately that the difference in potential energy per unit mass associated with the given altitude difference is negligible compared to the energies gained by burning both wood and acetylene. Hence, we get

$$\frac{E_\mathrm{d}}{E(\mathrm{CaC_2})} = \frac{H_\mathrm{d} - gh}{\frac{M(\mathrm{C_2H_2})}{M(\mathrm{CaC_2})}H(\mathrm{C_2H_2}) - gh} \approx \frac{M(\mathrm{CaC_2})H_\mathrm{d}}{M(\mathrm{C_2H_2})H(\mathrm{C_2H_2})} \doteq 0.72\,.$$

By the above specified criteria, it is more efficient to use acetylene as fuel.

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Problem FoL.18 ... bullet

In a dark abandoned hall, there are a robber and a policeman standing against each other by the opposite sides of the hall. Having heard steps echoing in the darkness, the criminal points *his gun horizontally on the unsuspecting cop and is about to shoot. What he doesn't know is that a potential difference $U = 10\,\mathrm{kV}$ was brought between the floor and the ceiling of the hall, thus creating a homogeneous field acting (vertically downwards) on the fired bullet, since the bullet acquires a charge q as it travels through the muzzle. Find the least charge-to-mass ratio q/m of the bullet such that it hits the floor before it reaches the policeman (where by m, we denote the mass of the bullet). The robber and the policemen are separated by a distance of $D = 100\,\mathrm{m}$, the ceiling is $d = 8\,\mathrm{m}$ above the floor and the bullet is fired horizontally with muzzle velocity $v_0 = 400\,\mathrm{m{\cdot}s^{-1}}$. Assume that the robber holds his gun $h = 1.5\,\mathrm{m}$ above the floor and that the acceleration due to gravity is $g = 9.81\,\mathrm{m{\cdot}s^{-2}}$.*

Mirek wanted to point out the importance of triboelectric effect.

Clearly, the force acting on a charge q in a homogeneous electric field of magnitude $E = U/d$ has magnitude $F_\mathrm{e} = qE = Uq/d$. If we also take into account the force $F_g = mg$ acting on a particle with mass m in a homogeneous gravitational field, the magnitude of total acceleration of the particle can be expressed as

$$g' = \frac{F_\mathrm{g} + F_\mathrm{e}}{m} = g + \frac{Uq}{md}\,.$$

Recalling that the range of a horizontally projected particle can be written as

$$D = \sqrt{\frac{2h}{g'}}v_0$$

and substituting for g' from above, we obtain

$$D = \sqrt{\frac{2h}{g + \frac{Uq}{md}}}v_0\,,$$

which can be solved for q/m to yield

$$\frac{q}{m} = \left(\frac{2hv_0^2}{D^2} - g\right)\frac{d}{U} \doteq 0.031\,\mathrm{C{\cdot}kg^{-1}}\,.$$

The bullet needs to have charge-to-mass ratio of at least $0.031\,\mathrm{C{\cdot}kg^{-1}}$.

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Problem FoL.19 ... watermelon

Náry was sitting on a swing, which was at rest. Suddenly, he felt like eating a watermelon, so other FYKOS guys immediately threw him one with mass $m_m = 20\,\mathrm{kg}$. Assuming that the speed of the watermelon was $v_m = 10\,\mathrm{m{\cdot}s^{-1}}$, find the angle by which the swing with Náry moves. Náry (approximated by a point particle) weighs $m_n = 80\,\mathrm{kg}$ and the swing rope is $L = 4\,\mathrm{m}$ long. Also, assume that the watermelon hit Náry (who managed to catch it) horizontally in the plane of rotation of the swing. State your answer in degrees. Gravitational acceleration is $g = 9.81\,\mathrm{m{\cdot}s^{-2}}$.

Lydka was eating a watermelon.

Conserving linear momentum, we can write

$$m_\mathrm{m} v_\mathrm{m} + m_\mathrm{n} v_\mathrm{n} = (m_\mathrm{m} + m_\mathrm{n})\, u\,,$$

where u is the speed of the combined system of Náry and the watermelon and $v_\mathrm{n} = 0\,\mathrm{m\cdot s^{-1}}$ is Náry's initial speed. Conserving mechanical energy, we have

$$\frac{1}{2}\,(m_\mathrm{m} + m_\mathrm{n})\, u^2 + 0 = 0 + (m_\mathrm{m} + m_\mathrm{n})\, hg\,,$$

where h is the maximum height, which Náry and the watermelon can reach (Náry's initial height is taken to be zero). These two expressions can be combined to give

$$h = \frac{(m_\mathrm{m} v_\mathrm{m})^2}{2\,(m_\mathrm{m} + m_\mathrm{n})^2\, g}\,.$$

So, the maximum deflection angle α of the swing from the vertical axis satisfies

$$\cos\alpha = \frac{L - h}{L} \quad \Rightarrow \quad \alpha \doteq 18.4^\circ\,.$$

Therefore, as a result of Náry catching his watermelon, the swing moves by 18.4°.

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Problem FoL.20 ... a superconducting one

Consider two superconducting capacitors with capacitances $C_1 = 10\,\mathrm{\mu F}$, $C_2 = 5.0\,\mathrm{\mu F}$ and a superconducting DC power supply with voltage $U = 10\,\mathrm{V}$. All these components are connected by superconducting wires. The superconductors are perfect, hence there is no resistance and therefore, Joule heating will not occur. First, let us connect the first capacitor to the DC supply and wait until the charge builds up. When it is fully charged, we disconnect the DC power supply and connect the second capacitor to the first one instead. Find the difference between the energy stored in the first capacitor alone and the total energy of the two connected capacitors.

Even the written exams can be educating for Mirek.

The charge on the first capacitor is $Q_1 = C_1 U$, so its energy is found to be

$$W_1 = \frac{Q_1^2}{2C_1} = \frac{1}{2} C_1 U^2 = 5 \cdot 10^{-4}\,\mathrm{J}\,.$$

After connecting the second capacitor and disconnecting the power supply, the total charge is conserved, but it is redistributed. The capacity of the two capacitors connected „in parallel“ is simply $C = C_1 + C_2$, so the total energy of the system is

$$W = \frac{Q_1^2}{2(C_1 + C_2)} \doteq 3.3 \cdot 10^{-4}\,\mathrm{J}\,.$$

Hence, the difference of the two energies is

$$\Delta W = W_1 - W = \frac{C_1 C_2 U^2}{2(C_1 + C_2)} \doteq 1.7 \cdot 10^{-4}\,\mathrm{J}\,.$$

In spite of the fact that Joule heating does not occur, the answer is not zero, because some energy is emitted in the form of electromagnetic waves emerging as a consequence of oscillating charge distribution.

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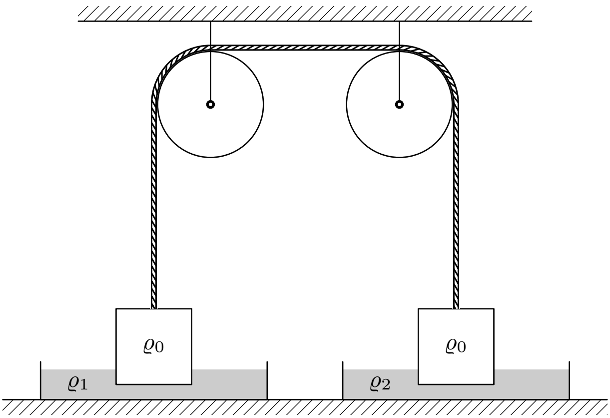
Problem FoL.21 ... buoyant swings with a pulley

Consider two identical cylinders (base radius $5\,\mathrm{cm}$, *height* $10\,\mathrm{cm}$ *and density* $\varrho_0 = 2.2\,\mathrm{g\cdot cm^{-3}}$*) in equilibrium, partially submerged in two containers filled with liquids of densities* $\varrho_1 = 1.0\,\mathrm{g\cdot cm^{-3}}$ *and* $\varrho_2 = 0.8\,\mathrm{g\cdot cm^{-3}}$ *(see the diagram below). Find the period of small vertical oscillations of this system about its equilibrium. Neglect the changes in the level of liquids in the containers. Neglect also the mass of the strings. The acceleration due to gravity is* $g = 9.81\,\mathrm{m\cdot s^{-2}}$. *Due to Janči.*



The x components of equations of motion for the two cylinders are

$$m\ddot{x} = -T - Sx\varrho_1 g$$

and

$$m\ddot{x} = T - Sx\varrho_2 g\,,$$

where m denotes the mass of one cylinder, S its horizontal cross-section and T is the tension in the string. Let us choose to measure the x coordinate of the first cylinder vertically downwards and the acceleration of the second cylinder vertically upwards. We then obtain

$$m\ddot{x} = -\frac{1}{2}Sx(\varrho_1 + \varrho_2)g\,,$$

which is of the form of an equation of simple harmonic motion with period

$$T = 2\pi\sqrt{\frac{2m}{S(\varrho_1 + \varrho_2)g}} = 2\pi\sqrt{\frac{2V}{gS}\frac{\varrho_0}{\varrho_1 + \varrho_2}}\,.$$

Since V/S is simply the height of one cylinder, we get $T \doteq 0.99\,\mathrm{s}$.

Interestingly, the period depends only on the height and density of the cylinders. This can be interpreted considering a possibility of superposition of such oscillators.

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Problem FoL.22 ... on the head

A new notice board needs a new nail (with length $l = 80\,\mathrm{mm}$), with which it should be nailed to the wall using our hammer with mass $m = 0.4\,\mathrm{kg}$. Five hits in total were needed to do this. During each hit, the speed of the hammer just before it hit the head of the nail was $v = 8.0\,\mathrm{m\cdot s^{-1}}$ (assume that upon hitting the nail, the hammer transfers all its kinetic energy into the nail). However, having nailed it down, we noticed that the board hangs askew. What is the magnitude of the force, with which we will need to pull the nail out of the wall? Assume that the mass of the nail is negligible. Also, assume that the magnitude of the frictional force is directly proportional to the length of the part of the nail, which is inside the board, and that it is the same during both pulling the nail out and nailing it in.

f(Aleš) was cleaning his notice board.

The work W needed to hammer the nail down is

$$W = 5\frac{mv^2}{2}\,.$$

Taking into account that the magnitude of the frictional force is directly proportional to the length of the part of the nail inside the board, W can also be expressed as

$$W = \frac{1}{2}F_\mathrm{o}l\,,$$

where F_o is the maximum magnitude of the frictional force (which is attained when the nail is fully inside the board).

Whence the magnitude of the frictional force (and so the magnitude of the force with which we will need to pull the nail out of the wall) is easily read off to be

$$F_\mathrm{o} = \frac{2W}{l} = \frac{5mv^2}{l} = 1\,600\,\mathrm{N}\,.$$

Force needed to pull the nail is equal to $1\,600\,\mathrm{N}$.

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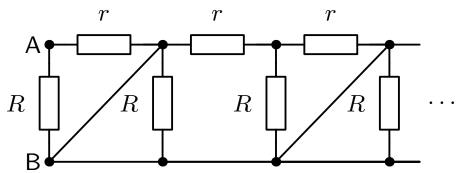
Problem FoL.23 ... an infinite circuit

Find the resistance between the terminals A *and* B *as shown on the diagram below. Put $R = 5\,\Omega$, $r = 1.5\,\Omega$ and neglect the resistance of leads.*

Xellos found solving the infinite resistor ladder too easy.



Everything apart from the two resistors which are closest to A is connected in parallel with a wire of zero resistance and therefore can be ignored. We are left with two resistors R and r in parallel which gives us the combined resistance as

$$\frac{Rr}{R+r} \doteq 1.15\,\Omega\,.$$

Note that the name of the question was misleading, we did not really need to consider any infinite series of resistors.

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Problem FoL.24 ... bowling with a physicist

Once, when Olda went to play bowling, an intriguing thing happened. He threw the bowling ball of mass $7.25\,\mathrm{kg}$ *and radius* $0.108\,\mathrm{m}$ *with speed* $12\,\mathrm{m\cdot s^{-1}}$ *and gave it such a spin that the ball stopped and stayed put at rest before reaching the skittles. What was the angular speed (in radians per second) which Olda had to give the ball for this to occur? Gravitational acceleration is* $9.81\,\mathrm{m\cdot s^{-2}}$*. Air resistance and rolling friction can be neglected.*

Organizers went bowling.

First we need to think about the mechanism which makes the ball stop. The only thing we were not told to neglect is the force due to friction between the ball and the floor whose magnitude we denote by F_T. Of course since mechanical energy is dissipated in all processes involving friction, we will not be able to make use of conservation of energy.

To solve the problem, we will revert to fundamental equations of Newtonian mechanics, namely the Newton's second law

$$F_\mathrm{T} = \frac{\mathrm{d}p}{\mathrm{d}t}$$

which in our case can be recast into

$$F_\mathrm{T} t = m v_0\,,$$

where v_0 is the initial speed, m mass of the ball and t the time from the moment the ball was thrown until it stops. Also, we can write down the rotational analogue of the Newton's second law, i.e.

$$F_\mathrm{T} r = \frac{\mathrm{d}L}{\mathrm{d}t} = \frac{\mathrm{d}J\omega}{\mathrm{d}t}$$

which, again, can be rewritten as

$$F_\mathrm{T} r t = J\omega_0\,,$$

where J is the moment of inertia of the ball, r its radius and ω_0 its initial angular speed.

Finally, eliminating t and substituting for the moment of inertia, we obtain

$$\omega_0 = \frac{F_\mathrm{T} r t}{J} = \frac{5F_\mathrm{T} r}{2mr^2}\frac{mv_0}{F_\mathrm{T}} = \frac{5v_0}{2r}\,.$$

Plugging in the numbers, we get $\omega_0 \doteq 280\,\mathrm{rad\cdot s^{-1}}$. A value this high could be anticipated by looking at the initial value of the translational kinetic energy.

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Problem FoL.25 ... waves and wavelets

As it known, propagation of surface waves on a lake is not significantly influenced by gravity in the short-wavelength limit. Nevertheless there is a strong effect of surface tension which needs to be considered. Using dimensional analysis, determine the angular frequency of surface waves as a function of surface tension σ, water density ϱ and the wave number $k = 2\pi/\lambda$, where λ is the wavelength. Assume that the dimensionless coefficient is equal to one in this case. Also, derive an expression for the group velocity and substitute the values $\sigma = 73\,\mathrm{mN\cdot m^{-1}}$, $\varrho = 1\,000\,\mathrm{kg\cdot m^{-3}}$, $\lambda = 2\,\mathrm{cm}$. *Mirek trying to catch short waves.*

We aim to write down a relation of form $\omega = \sigma^\alpha \varrho^\beta k^\gamma$. Decomposing the unit of newton as $\mathrm{kg\cdot m\cdot s^{-2}}$, we obtain the following system of linear equations for α, β, γ

$$\begin{aligned} 0 &= \alpha + \beta\,,\\ 0 &= -3\beta - \gamma\,,\\ -1 &= -2\alpha\,. \end{aligned}$$

We can immediately read off that $\alpha = 1/2$. Substituting this into the remaining two equations, we get $\beta = -1/2$ and $\gamma = 3/2$. Hence the angular frequency is given by (putting the dimensionless coefficient equal to 1, as we are told)

$$\omega = \sqrt{\frac{\sigma k^3}{\varrho}}\,.$$

Using the definition of the group velocity, we can write

$$v_\mathrm{g} = \frac{\partial \omega}{\partial k} = \frac{3}{2}\sqrt{\frac{\sigma k}{\varrho}}\,.$$

Substituting $k = 2\pi/\lambda$ and the rest of data given, we get

$$v_\mathrm{g} = \frac{3}{2}\sqrt{\frac{2\pi\sigma}{\varrho\lambda}} \doteq 0.23\,\mathrm{m\cdot s^{-1}}\,,$$

which is the magnitude of group velocity.

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Problem FoL.26 ... rolling cylinder

As a proper engineer, Dominika often indulges herself with rolling homogeneous cylinders down a hill. At one such occasion (having a moment of clarity) she came up with an alternative method of determining the number of turns the cylinder completes before it falls off an inclined plane. First, she measured its mass ($m = 123.8\,\mathrm{g}$) and also its moment of inertia $J = 1.807\cdot 10^{-5}\,\mathrm{kg\cdot m^2}$. It is widely known that Dominika's inclined plane is $l = 1.02\,\mathrm{m}$ long and that it makes an arbitrary angle α with horizontal. How many turns (real number, to three significant figures) does the cylinder complete before it reaches the end of the inclined plane? Assume that the cylinder rolls precisely from one end of the plane to the other and that it moves without slipping in the direction of maximum slope.

Karel wondered about Dominika's future in the field of engineering.

Moment of inertia of a solid homogeneous cylinder is $J = mr^2/2$. Knowing J and also the mass m we can calculate the radius of Dominika's cylinder as

$$r = \sqrt{\frac{2J}{m}} \doteq 1.71\cdot 10^{-2}\,\mathrm{m}\,.$$

The number of turns N completed can then be obtained as the ratio of total distance l covered and circumference $o = 2\pi r$ of the cylinder. Thus

$$N = \frac{l}{o} = \frac{l}{2\pi r} = \frac{l}{2\pi}\sqrt{\frac{m}{2J}} \doteq 9.50\,.$$

Hence the cylinder completes 9.50 turns before it reaches the end of the inclined plane.

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Problem FoL.27 ... Zdeněk's sphere

As a proper soon-to-be-engineer, Zdeněk (like Dominika) likes to roll various objects down an inclined plane. Nevertheless, unlike Dominika, who is rather more into cylinders, Zdeněk prefers to play with his sphere. He is not a fan of ordinary experimental methods, so in order to find out the density of his sphere, he decided to measure its mass $m = 123.8\,\mathrm{g}$ (which is incidentally the same as that of Dominika's cylinder) and its moment of inertia $J = 3.48\cdot 10^{-5}\,\mathrm{kg\cdot m^2}$. Your task is to help Zdeněk with his calculation and determine the density of his solid homogeneous sphere.

Karel fantasizing about BUT…

Moment of inertia of a solid homogeneous sphere with mass a and radius r is $J = 2mr^2/5$. The radius can then be expressed as

$$r = \sqrt{\frac{5J}{2m}}\,.$$

Density ϱ of the sphere can be found as the ratio of its mass m and its volume $V = 4\pi r^3/3$, i.e.

$$\varrho = \frac{m}{V} = \frac{m}{\frac{4}{3}\pi r^3} = \frac{3m}{4\pi}\left(\frac{2m}{5J}\right)^{\frac{3}{2}} = \frac{3m^{\frac{5}{2}}}{\sqrt{2}5^{\frac{3}{2}}\pi J^{\frac{3}{2}}} \doteq 1.59\cdot 10^3\,\mathrm{kg\cdot m^{-3}}\,.$$

Hence the density of Zdeněk's sphere is $1\,590\,\mathrm{kg\cdot m^{-3}}$.

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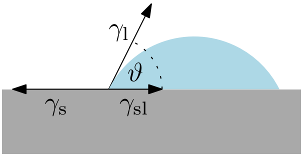
Problem FoL.28 ... energetic surface

What is the contact angle (in degrees) of the interface of air, water (surface tension $\gamma_l = 74\,\mathrm{mN{\cdot}m^{-1}}$) and an unknown solid with surface energy (sc. the solid-air interfacial energy) $\gamma_s = 53\,\mathrm{mN{\cdot}m^{-1}}$? The solid-water interfacial energy is $\gamma_{sl} = 25\,\mathrm{mN{\cdot}m^{-1}}$.

A by-product of Terka's work on a FYKOS experimental task.

The interfacial energy (that is to say the surface tension – for a pure liquid, these two notions coincide) gives the magnitude of a force acting on a line element on the interface divided by the length of that line element. The vector of this force, which is perpendicular to the line element under consideration, lies in the plane which is tangent to the interface.



A droplet of water placed onto our mysterious material takes a steady axisymmetric form. In particular, this means that the horizontal components of forces along the line of contact must sum to zero. Mathematically, this is expressed by what is normally referred to as the Young's equation

$$\gamma_\mathrm{sl} + \gamma_\mathrm{l} \cos\vartheta = \gamma_\mathrm{s}\,,$$

where ϑ denotes the contact angle. Whence

$$\vartheta = \arccos \frac{\gamma_\mathrm{s} - \gamma_\mathrm{sl}}{\gamma_\mathrm{l}} \doteq 68^\circ\,.$$

Hence the contact angle is 68°. This is consistent with our mysterious material being e.g. nylon.

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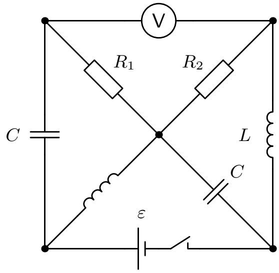
Problem FoL.29 ... pyramid's circuits vol. 2

In the circuit shown below we first put the switch on and wait for the currents to come to a steady state. Then we put the switch back off. What is the magnitude of the potential difference (in volts) measured by the voltmeter immediately after the switch was turned off? Take the capacitors and coils to be ideal and assume that the voltmeter provides an infinite resistance. Put $\varepsilon = 1\,\mathrm{V}$ and $R_1 = 3R_2$.

Xellos found out about the Estonian-Finnish Olympiad in Physics.



We'll calculate currents through the resistors from the top vertex of the pyramid (junction in the center of the picture).

When the switch is on and the currents reach steady state, the capacitors and voltmeter can be thought of as elements with infinite resistance. On the other hand, the coil behaves like a conductor with no resistance. Hence the currents through R_1 and R_2 are $I_1 = 0$ and $I_2 = \varepsilon/R_2$ respectively.

The properties of capacitors and coils are such that it is not possible to create a temporal discontinuity in potential difference across a capacitor and similarly with current through a coil. Therefore, after the switch is put off, there is a current $I_2 = \varepsilon/R_2$ through the resistor R_2. Also, the current through the resistor R_1 is now $I_3 = I_2$, so the potential difference across the two resistors is

$$|R_1 I_3 - R_2 I_2| = 2\varepsilon = 2\,\mathrm{V}\,.$$

Hence the reading shown by the voltmeter is 2 V.

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Problem FoL.30 ... gassed joules

Náry once passed by a pressure vessel containing sulphur dioxide to which a number of meters were attached and some data were printed on its mantle. Namely, he managed to read off that the volume of the vessel was $V = 1\,\mathrm{m}^3$ and that the mass of an empty vessel would be $m = 100\,\mathrm{kg}$. The meters were showing the temperature of the gas inside, its pressure $p = 5.3\,\mathrm{MPa}$ as well as the combined mass of the gas and the vessel $M = 152\,\mathrm{kg}$. However, being myopic, Náry was unable to read off the value of the temperature. Luckily, he had 4 hours of free time so he set off to calculate the temperature assuming there was an ideal gas inside the vessel. Deeming the value he obtained too low and having another 6 hours free, he decided to recalculate the temperature under the assumptions of van der Waals theory. What is the magnitude of the difference of the two results he got (in kelvins)?

f(Aleš) remembered his Thermodynamics example classes.

We want to calculate the difference $\Delta T = T_\mathrm{v} - T_\mathrm{i}$ of the temperatures yielded by the two models. Hence we need to calculate the temperature in each model first, i.e. we need to find an expression for the temperatures T_v and T_i of a van der Waals and an ideal gas, respectively. Let us start with ideal gas, just as Náry did. From the Ideal Gas Law

$$pV = nRT_\mathrm{i}\,,$$

the temperature T_i can be easily expressed as

$$T_\mathrm{i} = \frac{pV}{nR}\,.$$

In order to find out the temperature of a van der Waals gas, we need to know what the van der Waals equation of state looks like. We have

$$\left(p + \frac{a}{v^2}\right)(v - b) = RT_\mathrm{v}\,,$$

where $v = V/n$ is the molar volume. We then obtain

$$T_\mathrm{v} = \frac{1}{nR}\left(p + \frac{n^2 a}{V^2}\right)(V - nb)\,.$$

It remains to subtract the two derived expressions and to calculate the amount n of gas inside the vessel as

$$n = \frac{M - m}{M_{\mathrm{SO}_2}}\,,$$

where M_{SO_2} is the molar mass of sulphur dioxide. Hence

$$\Delta T = \frac{M_{SO_2}}{(M-m)\,R}\left[\left(p + \frac{(M-m)^2 a}{V^2 M_{SO_2}^2}\right)\left(V - \frac{M-m}{M_{SO_2}}b\right) - pV\right].$$

Finally we recall that the universal gas constant is $R = 8.314\,\mathrm{J\cdot K^{-1}\cdot mol^{-1}}$, that the coefficients of the van der Waals equation for sulphur dioxide are $a = 0.680\,\mathrm{J\cdot m^3\cdot mol^{-2}}$, $b = 5.64\cdot10^{-5}\,\mathrm{m^3\cdot mol^{-1}}$ and that the molar mass of sulphur dioxide is $M_{SO_2} = 6.41\cdot10^{-2}\,\mathrm{kg\cdot mol^{-1}}$. We then obtain $\Delta T \doteq 28\,\mathrm{K}$.

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Problem FoL.31 ... elusive lens

During a clean-up day in the FYKOS room, a thin lens together with a circular-aperture light source with diameter $d = 5\,\mathrm{mm}$ were found. First consider placing the lens in a distance $s_1 = 12\,\mathrm{cm}$ from a screen. If we then put the source in a large distance from both the screen and the lens, we will observe a point-like image on the screen. If on the other hand we place the lens and the source in a distance s_2 and l_2 respectively from the screen, we will observe a sharp circular image with diameter $d_2 = 2\,\mathrm{cm}$. Find the distance l_2 (in cm).

Outrageously, Xellos prefers doing experiments rather than helping with clean- up.

It is clear from the first case that the focal length of the lens is $f = s_1$. In the second case an image is projected onto the screen with linear magnification

$$Z = \frac{d_2}{d} = \frac{s_2}{l_2 - s_2} = 4\,,$$

whence

$$s_2 = l_2 \frac{Z}{1+Z}\,.$$

Finally, using the thin lens formula, we have

$$\frac{1}{s_2} + \frac{1}{l_2 - s_2} = \frac{1}{f}\,,$$

$$\frac{1+Z}{Zl_2} + \frac{1+Z}{l_2} = \frac{1}{f}\,,$$

$$\frac{(1+Z)^2}{Zl_2} = \frac{1}{f}\,,$$

$$l_2 = \frac{(1+Z)^2}{Z} f \doteq 75\,\mathrm{cm}\,.$$

Hence in the second case the source was placed 75 cm from the screen.

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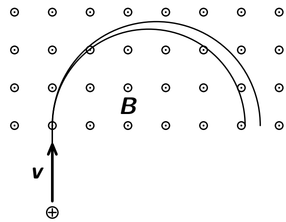
Problem FoL.32 ... playing in the field

Maťko and Kubko, two neon isotopes with mass numbers $A_M = 20$ and $A_K = 22$, came up with a new game, where Maťko simply tries to copy all moves of Kubko. Carrying a charge of $+e$, Kubko entered a homogeneous magnetic field with magnitude $B = 0.24\,\mathrm{T}$ in the direction perpendicular to the field lines. Maťko, who also carries a charge of $+e$, entered the field at the same point and in the same direction as Kubko did. Find the distance d of the points where the respective isotopes left the field, provided they both have the same kinetic energy $E_K = 6.2 \cdot 10^{-16}\,\mathrm{J}$. Take the values of the atomic mass unit and the elementary charge to be $m_u = 1.660\,57 \cdot 10^{-27}\,\mathrm{kg}$ and $e = 1.602 \cdot 10^{-19}\,\mathrm{C}$. *Lydka likes to play with fridge magnets.*



Let us denote by r_M and r_K the radii of circular trajectories followed by Maťko and Kubko. The constraints of the problem then imply $d = 2|r_\mathrm{K} - r_\mathrm{M}|$. First, let us find a general expression for the radius of the trajectory followed by a charged particle in a homogeneous magnetic field. Equating the magnitude of the centripetal force $F_\mathrm{c} = mv^2/r$ with the magnitude of the Lorentz force $F_\mathrm{m} = Bev\sin\alpha$ (where in our case we actually have $\alpha = 90°$) we get

$$r = \frac{mv}{Be}\,.$$

Next,

$$E_\mathrm{K} = \frac{mv^2}{2} \quad \Rightarrow \quad v = \sqrt{\frac{2E_\mathrm{K}}{m}}\,.$$

Also, the mass of a particle with mass number A clearly satisfies $m = Am_\mathrm{u}$. Combining the two expressions for r and v above gives

$$r = \frac{\sqrt{2Am_\mathrm{u}E_\mathrm{K}}}{Be}$$

and so d is as follows

$$d = 2\frac{\sqrt{2A_\mathrm{K}m_\mathrm{u}E_\mathrm{K}}}{Be} - 2\frac{\sqrt{2A_\mathrm{M}m_\mathrm{u}E_\mathrm{K}}}{Be} = 0.016\,3\,\mathrm{m}\,.$$

Thus wanted distance of the points is 0.0163 m.

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Problem FoL.33 ... planet Fykosia

Fykosia is an as yet undiscovered planet of our Solar System. Find the ratio a/r (where r is the radius of Fykosia and a is its distance from the Sun) and submit your answer as a multiple of π. Assume that the trajectories along which the planets orbit the Sun are circles and that the Sun lies in their common centre. The orbital period of Fykosia is $T = 3\sqrt{3}\,\mathrm{yr}$ and its density is $\varrho = 5\,000\,\mathrm{kg{\cdot}m^{-3}}$. Also, assume that the magnitude of the force due to gravity acting on a body with mass $m = 1\,\mathrm{kg}$ placed on the surface of Fykosia is $F_G = 69\,\mathrm{N}$. The gravitational constant is $G = 6.67 \cdot 10^{-11}\,\mathrm{N{\cdot}m^2{\cdot}kg^{-2}}$. *Fykosie, my homeland!*

On one hand, using the third Kepler's law we have

$$\frac{T^2}{a^3} = \frac{T_Z^2}{a_Z^3}\,,$$

where $T_Z = 1\,\mathrm{yr}$ is the orbital period of Earth and $a_Z = 1\,\mathrm{AU}$ is the distance between the Earth and the Sun. Whence

$$a = \left(\frac{T}{T_Z}\right)^{2/3} a_Z\,.$$

On the other hand, the Newton's law of gravity says that

$$F_G = G\frac{Mm}{r^2}\,,$$

where M is the mass of Fykosia, for which we can write

$$M = V\varrho = \frac{4}{3}\pi\varrho r^3\,.$$

Substituting into the Newton's law and handling the units carefully, we finally arrive at

$$r = \frac{3F_G}{4\pi\varrho G m}\,.$$

Therefore the ratio is $a/r \doteq 2\,900\pi$.

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Problem FoL.34 ... feather-light

Imagine yourself in the role of an observer situated in the direction of north pole of a galaxy called Fykopaedia, which you observe rotating clockwise. Once you have managed to identify what appears to be the galactic centre, you also spot a star orbiting the centre along a circular trajectory with radius $r = 7\,500\,\mathrm{pc}$ at the very edge of the galaxy. Having obtained a spectrum of this star, you notice that a hydrogen line with laboratory wavelength $\lambda_{emit} = 587.49\,\mathrm{nm}$ is transverse Doppler-shifted to $\lambda_{obs} = 589.89\,\mathrm{nm}$. Your task is now to find the mass of Fykopaedia in the units of solar mass. You should disregard any effects due to possible presence of dark matter or dark energy. Also, assume that the observed spectral shift is solely due to transverse Doppler effect and that the mass distribution in the galaxy is spherically symmetric.

f(Aleš) complaining about his heavy bag.

Given the distance of the star from the galactic centre and its speed, it is easy to infer its orbital period. We can then apply the Third Kepler's law to obtain the mass of the galaxy, i.e.

$$M = \frac{4\pi^2 r^3}{GT^2} = \frac{rv^2}{G}\,,$$

where v is the orbital speed and G is the gravitational constant. Hence we are left to find the orbital speed, for which we may write

$$\frac{\lambda_{obs}}{\lambda_{emit}} = \frac{1}{\sqrt{1 - v^2/c^2}}\,,$$

which follows from the formalism of transverse Doppler effect. The speed v can then be expressed as

$$v^2 = c^2 \left(1 - \left(\frac{\lambda_{\mathrm{emit}}}{\lambda_{\mathrm{obs}}}\right)^2\right) .$$

Substituting into the expression for the mass derived earlier, we get

$$M = \frac{rc^2 \left(1 - \left(\frac{\lambda_{\mathrm{emit}}}{\lambda_{\mathrm{obs}}}\right)^2\right)}{G} .$$

Numerically $M \doteq 2.5 \cdot 10^{45}\,\mathrm{kg} \doteq 1.3 \cdot 10^{15} M_{\mathrm{S}}$, where the values $M_{\mathrm{S}} = 1.99 \cdot 10^{30}\,\mathrm{kg}$ and $G = 6.67 \cdot 10^{-11}\,\mathrm{m^3 \cdot kg^{-1} \cdot s^{-2}}$ were used.

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Problem FoL.35 ... an accelerated one

Consider a proton travelling inside a synchrotron with a total energy of $E_{\mathrm{p}} = 1.00\,\mathrm{GeV}$. *The proton collides with an alpha particle with a total energy of* $E_\alpha = 3.75\,\mathrm{GeV}$. *Given the rest masses of a proton and an alpha particle* $m_{\mathrm{p}} = 0.938\,\mathrm{GeV}/c^2$ *and* $m_\alpha = 3.727\,\mathrm{GeV}/c^2$, *respectively, determine the ratio* v_{p}/v_α *of the speeds of the particles before they collide.*

Mirek likes getting results which resemble fundamental mathematical constants.

It follows from the principles of the theory of special relativity that total energy of a particle is given by

$$E = \gamma mc^2 ,$$

where the speed of the particle is hidden inside the Lorentz factor

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$

and thus can be expressed as

$$v = c\sqrt{1 - \left(\frac{mc^2}{E}\right)^2} .$$

Therefore the ratio v_{p}/v_α satisfies

$$\frac{v_{\mathrm{p}}}{v_\alpha} = \frac{E_\alpha}{E_{\mathrm{p}}} \sqrt{\frac{E_{\mathrm{p}}^2 - (m_{\mathrm{p}}c^2)^2}{E_\alpha - (m_\alpha c^2)^2}} \doteq 3.13 .$$

The ratio of velocities is equal to 3.13.

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Problem FoL.36 ... resistance upside down

We want to build a circuit with total resistance $R = (57\,688/1\,807)\,\Omega$, but we only have resistors with resistance $r = 1\,\Omega$ at our disposal. On top of that, we want to build our circuit so that in each step (starting with one resistor) we connect only one resistor to an already existing part of the circuit, either in series or in parallel. What is the smallest number of resistors we need to reach our goal?

Xellos recalled a physically motivated problem from a programming competition.

Starting with a resistance $R/r = a/b$ we can make it either into a resistance $(a + b/b$ by connecting one resistor in series, or, into a resistance $a/(b + a)$ by connecting one resistor in parallel. In other words, resistance a/b can be built starting with either $(a - b)/b$ (if $a \geq b$) or $a/(b-a)$ (if $b > a$) and with nothing else. Therefore the number of resistors needed to build a/b is unique and can be calculated by subtracting the numerator from the denominator or vice versa. Hence we find out that getting to $a/b = 0$ takes 68 steps.

There is a faster approach, which makes use of the fact that a/b can be built using the same number P of steps as b/a, hence

$$P\left(\frac{a}{b}\right) = P\left(\frac{b}{a}\right) \,,$$

and for $a \geq b$ have

$$P\left(\frac{a}{b}\right) = \left\lfloor\frac{a}{b}\right\rfloor + P\left(\frac{a \bmod b}{b}\right) \,.$$

Hence the answer is $31 + 1 + 12 + 3 + 2 + 19 = 68$.

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Problem FoL.37 ... nomen omen

During a meeting of the greatest villains of all time, The Emperor was laughed at: his Death Star has a word "star" in its name, yet it does not shine. He could not possibly let such insolence pass and decided to fix the problem. His Death Star now emits light as if it was a gray body with emissivity $\varepsilon = 0.8$ across all wavelengths. (Emissivity of a body is defined to be the ratio of intensity of radiation emitted by the body to the intensity of radiation emitted by a black body at the same temperature.) The radius of the Death Star is $r = 450\,\mathrm{km}$ and an inspecting spaceship, being at a distance of $d = 10\,000\,\mathrm{km}$ from the Star, measured intensity $I = 33\,\mathrm{kW{\cdot}m^{-2}}$ of radiation coming from the Star. What is the wavelength (in nanometres) at which the Death Star radiates most of its energy? Stefan-Boltzmann constant is $\sigma = 5.67 \cdot 10^{-8}\,\mathrm{W{\cdot}m^{-2}{\cdot}K^{-4}}$, Wien's displacement constant is $b = 2.90 \cdot 10^{-3}\,\mathrm{m{\cdot}K}$.

Mirek lives by the philosophy that there is never enough Star Wars themed problems.

Intensity of radiation decreases with the square of the distance. Therefore, we can use the Stefan-Boltzmann law to find that

$$I(d/r)^2 = \varepsilon\sigma T^4 \,.$$

We then use Wien's displacement law to calculate the wavelength at which the Death Star radiates most of its energy. We obtain

$$\lambda_{\max} = \frac{b}{T} = \frac{b}{\sqrt[4]{d^2 I/(\varepsilon\sigma r^2)}} \doteq 666\,\mathrm{nm} \,.$$

The Death Star radiates most of its energy at wavelength 666 nm.

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Problem FoL.38 ... pendulum

Once upon a time there lived an infinite superconducting horizontal plane with surface mass density $\sigma = 2 \cdot 10^{10}\,\mathrm{kg{\cdot}m^{-2}}$ *and a pivot placed* $d = 5\,\mathrm{m}$ *above the plane. And so it came to pass that somebody came up with the idea to suspend an electron on a string with length* $l = 1\,\mathrm{m}$ *from the pivot. Find the period of small oscillations (in seconds) of the electron about its equilibrium.*

Assume that charge is free to move within the plane arbitrarily fast. Neglect any effects due to magnetic field. *Xellos walking an electron on a dog lead.*

We think of the electron as being a point charge $-e$ with mass m_e. There are two forces acting on it, both in the direction downwards perpendicular to the plane: electrostatic force with magnitude F_e and gravitational force $F_\mathrm{g} = m_\mathrm{e} g$.

Gauss's law of electrostatics may be (in an analogy) used to determine the magnitude of the acceleration due to gravity g. First, the field intensity is defined as a force per unit mass, which directly corresponds to acceleration in the case of gravitational field. Next, comparing Coulomb's and Newton's inverse square laws for fields due to point sources, we find that the factor of $1/\varepsilon_0$ needs to be replaced by $4\pi G$. Hence the Gauss's law of gravity can be written as

$$\oint_{\partial V} g\,\mathrm{d}S = 4\pi G m\,,$$

where m is total mass enclosed by volume V. Thus, choosing a suitable Gaussian surface, we obtain

$$g = 2\pi G\sigma\,.$$

Next, we use the method of images to find the magnitude F_e of the electrostatic force acting on the electron: given a charge $-e$ situated in a distance h above a conducting plane, the force acting on the charge is the same as if the plane was replaced by a mirror charge $+e$ situated in a distance h below the plane. Coulomb's law then gives

$$F_\mathrm{e} = \frac{e^2}{4\pi\varepsilon_0 (2h)^2}\,.$$

The distance $h \approx d - l$ between the electron and the plane stays roughly constant during the period of oscillations (which are assumed to be small). We can therefore conclude that there is a constant force with magnitude $F = F_\mathrm{e} + F_\mathrm{g} = 1.12 \cdot 10^{-29}\,\mathrm{N}$ acting on the electron. Next, upon disturbing the electron form its equilibrium sideways be an angle α, there appears a restoring torque $\tau \approx Fl\alpha$. Finally, given that the moment of inertia of the electron relative to the pivot is clearly $I = m_\mathrm{e} l^2$, the angular frequency ω of the oscillations satisfies

$$\omega^2 = \frac{4\pi^2}{T^2} = \frac{\tau}{I\alpha} = \frac{F}{m_\mathrm{e} l}\,,$$

whence $T = 2\pi\sqrt{m_\mathrm{e} l / F} \doteq 1.8\,\mathrm{s}$.

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Problem FoL.39 ... magnetic focusing

Assume that we fire a beam of protons from a given location on the x axis. In the beam, every proton has a speed of $v = 10^5\,\mathrm{m{\cdot}s^{-1}}$, whose direction deviates from that of the x axis by at most $\Delta\alpha = 1^\circ$. Assume that there is an electric field pointing in the x direction with magnitude $E = 12\,\mathrm{N{\cdot}C^{-1}}$, together with a magnetic field with magnitude $B = 1\,\mathrm{mT}$, also pointing in the x direction. The protons will first come back to the x axis at a distance from the firing point which lies in an interval of the form $(L, L + \Delta L)$. Find the ratio $(\Delta L)/L$. Do not take into account the protons fired precisely along the x axis. *Xellos was playing Bang!*

Consider a single particle of mass m and charge q, whose direction deviates from the x-axis by a small angle α. Its velocity in the direction perpendicular to x-axis is $v_\perp = v \sin\alpha$, which is a constant. Therefore the particle will move along a circular path of radius

$$R = \frac{v_\perp m}{qB}\,,$$

and it will intersect the x-axis again in time

$$T = \frac{2\pi R}{v_\perp} = \frac{2\pi m}{qB}\,.$$

The initial velocity in the direction of the x-axis is $v_\parallel = v\cos\alpha$ and the acceleration is $a = Eq/m$, so in the time T the particle travels in the direction of x-axis distance

$$L = v_\parallel T + \frac{Eq}{2m}T^2 = \frac{2\pi m v_\parallel}{qB} + \frac{2\pi^2 mE}{qB^2}\,.$$

It's obvious that $\Delta L = L(\alpha = 0) - L(\alpha = \Delta\alpha)$ and since the angle α is small, i.e. $\alpha \leq \Delta\alpha \ll 1$, we can use the approximation $\cos\alpha \approx 1 - \alpha^2/2$ and write

$$\Delta L \approx \frac{\pi m v (\Delta\alpha)^2}{qB}\,.$$

A less accurate approximation $\cos\alpha \approx 1$ yields

$$L \approx \frac{2\pi m v}{qB} + \frac{2\pi^2 mE}{qB^2}\,,$$

$$\frac{\Delta L}{L} \approx \frac{(\Delta\alpha)^2}{2 + \frac{2\pi E}{vB}} \doteq 1.1 \cdot 10^{-4}\,,$$

($\Delta\alpha$ is expressed in radians). The ratio $\Delta L/L$ equals $1.1 \cdot 10^{-4}$.

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Problem FoL.40 ... slingshooty

The Czech mythical genius Jára Cimrman is fabled for his construction of the so called Čakovice slingshot. Being curious, we loaded the slingshot with a projectile and fired. Having grossly underestimated its power, not only did we find that the projectile did not land in Čakovice (now a district of Prague), we discovered that it basically did not land at all. It went through *the Earth's atmosphere, across the Solar system, all the way out of the Galaxy, even escaping the local group, passing through the Virgo cluster... until it reached the very borders of the visible universe and ended up in an alternative one.*

The physical laws of this alternative universe are quite similar to those of our universe, except that the gravitational force between the projectile and rest of the matter in this alternative universe, while still derived from the Newton's law, acts repulsively. Consider the projectile approaching one of local stars. If there was no interaction between the projectile and the star, the projectile would miss the star at distance $b = 10^{10}\,\mathrm{m}$. However, as a result of the above described interaction, the trajectory of the projectile will be deflected. Assuming that the mass of the projectile is $m = 1\,\mathrm{kg}$, that the mass of the star is $M = 10^{30}\,\mathrm{kg}$ and that the speed of the projectile far away from the star is $v_0 = 10^5\,\mathrm{m{\cdot}s^{-1}}$, find the minimum separation of the projectile and the star (both of which are to be regarded as point particles) during their interaction. The gravitational constant is $G = 6.67\cdot 10^{-11}\,\mathrm{m^3{\cdot}kg^{-1}{\cdot}s^{-2}}$.

Mirek applied Rutherford's experiment in the macroworld.

We will approach the problem as if it was the Rutherford's gold foil experiment. In a planetary system angular momentum and total mechanical energy are always conserved. It is convenient to express these two conserved quantities in polar coordinates. We will set them equal to their initial values

$$L = mv_0 b = mr^2\dot{\varphi}\,,$$
$$E = \frac{1}{2}mv_0^2 = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\varphi}^2) + \frac{GMm}{r}\,.$$

Now we express the time derivative $\dot{\varphi}$ from the first equation and plug it into the second equation. After some algebraic manipulation we arrive at

$$v_0^2 = \left(\dot{r}^2 + \frac{L^2}{m^2r^2}\right) + \frac{2GM}{r}\,.$$

The projectile reaches its closest point to the sun when the radial distance r is minimal, which implies $\dot{r} = 0$ and the equation simplifies to

$$v_0^2 - \frac{2GM}{r} - \frac{L^2}{m^2r^2} = 0\,.$$

It remains to plug in the initial value of angular momentum and the minimum distance can be obtained by solving the quadratic equation in r. The only physically meaningful root is

$$r_{\mathrm{min}} = \frac{GM}{v_0^2} + \sqrt{\frac{G^2M^2}{v_0^4} + b^2} \doteq 1.87\cdot 10^{10}\,\mathrm{m}\,.$$

The minimum distance is $1.87\cdot 10^{10}\,\mathrm{m}$.

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Problem FoL.41 ... little gifts

Having left the student housing, Mirek's roommate left behind a number of interesting objects, amongst them a $10\,\mathrm{m}$ *long rope,* $170\,\mathrm{cm}$ *long iron rod and a bag full of cobble stones. Now, let's consider the following situation: first, let's tie the rope to one end of the rod and fix the bag with cobble stones to the same end. Then throw the rod with the bag out of the window, while holding the other end of the rope firmly. The rod lands on the pavement (which is* $d = 11\,\mathrm{m}$ *below the window) in such a way that its free end rests on the pavement exactly below the point where the other end of the rope is held. Assuming that the rod can rotate freely about the end on which it rests, but that the end itself cannot move (there is an infinite friction between the pavement and the rod), find the tension in the rope in equilibrium. The bag with cobble stones has mass* $m_k = 8\,\mathrm{kg}$*, the rod weighs* $m_t = 5\,\mathrm{kg}$ *and the stiffness of the rope is* $k = 200\,\mathrm{kg{\cdot}s^{-2}}$*. Neglect the mass of the rope.* *And don't forget to fix the road afterwards! (Mirek).*

Let us denote by L and l the length of the rod and the length of the rope, respectively. Define

$$M = \frac{m_\mathrm{t}}{2} + m_\mathrm{k}\,.$$

The total potential energy of the system is then given by

$$V = MgL\cos\alpha + \frac{k}{2}\left(\sqrt{L^2 + d^2 - 2dL\cos\alpha} - l_0\right)^2\,,$$

where l_0 is the rest length of the rope. Clearly, the first term in the above equation has the meaning of gravitational potential energy whereas the second term is nothing but the elastic potential energy $k(\Delta l)^2/2$. The total potential energy must have a minimum at equilibrium, hence

$$\frac{\mathrm{d}V}{\mathrm{d}\alpha} = 0\,.$$

Differentiating the above expression for V and equating the result to zero yields

$$l = \frac{l_0}{1 - Mg/kd}\,,$$

which we, in turn, substitute into the Hooke's law. We obtain

$$F = k(l - l_0) = \frac{kl_0}{kd/Mg - 1} \doteq 98.2\,\mathrm{N}\,.$$

The tension in the rope is $F \doteq 98.2\,\mathrm{N}$.

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Problem FoL.42 ... van de Graaff

Consider a conducting sphere with radius $R = 7\,\mathrm{cm}$ *charged to a potential* $\varphi = -5\,\mathrm{V}$ *(taken relative to a point at infinity) and covered with a thin layer of dielectric. The sphere is placed into a chamber with ionised helium* He^+ *where we assume that only the ions with negligible kinetic energy stick to the surface of the sphere. Finally, the sphere is moved into an evacuated chamber with volume* $V = 10\,\mathrm{m}^3$ *and quickly discharged. Find the equilibrium temperature (in kelvins) of the ion gas thus created in the chamber.* *Xellos loves electroshocks.*

The electrostatic potential on the surface of a homogeneously charged sphere is equal to the potential of a point charge $-Q$ placed in the center of the sphere

$$\varphi = -\frac{Q}{4\pi\varepsilon_0 R}\,.$$

The ions which stick to the sphere have the same magnitude of charge, so the sphere appears to be neutral and does not attract more ions.

After discharging, the force holding the ions to the surface of the sphere will cease to exist and all the ions will spread into the surrounding environment. The container is reasonably big, so we can safely assume that the potential energy will be transformed into kinetic energy. Now we will recall the equipartition theorem

$$E_\mathrm{k} = \frac{3}{2}k_\mathrm{B}T\,.$$

Therefore we need to compute electrostatic energy of one ion. That can be done for example by increasing the radius of the sphere with a charge Q and then computing the work done by the electric force on one ion, which is in fact equal to the difference of the energies with the electrostatic energy at infinity equal to zero.

As the next step we compute the electrical intensity acting upon a infinitesimal surface $\mathrm{d}S$ of continuously distributed charge on the sphere with the radius r. The intensity at the surface of the sphere, using Gauss's law, is

$$E_\mathrm{g} = \frac{Q}{4\pi\varepsilon_0 r^2}\,.$$

From this result we must subtract the intensity E_s from the surface $\mathrm{d}S$ itself. The chosen surface is small, therefore we can treat it as a short cylinder and find (Gauss's law again)

$$E_\mathrm{s} = \frac{Q}{8\pi\varepsilon_0 r^2}\,,$$

the force acting upon charged ion is

$$F = e(E_\mathrm{g} - E_\mathrm{s}) = \frac{eQ}{8\pi\varepsilon_0 r^2}\,.$$

Work is obtained as the integral

$$W = \int_R^\infty F\mathrm{d}r = \frac{eQ}{8\pi\varepsilon_0 R}\,,$$

and the temperature is

$$T = \frac{2W}{3k_B} = \frac{eQ}{12\pi\varepsilon_0 R k_B} = \frac{-e\varphi}{3k_B} \doteq 19\,300\,\mathrm{K}\,.$$

The gas in the container will have temperature $T \doteq 19\,300\,\mathrm{K}$.

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Problem M.1 ... electrical shopping cart

Being incredibly lazy, the organisers of FYKOS decided to construct a small self-propelled cargo vehicle. It remains to fine tune the engine. Having a total mass of $m = 70\,\mathrm{kg}$, the vehicle is known to accelerate with $a = 1\,\mathrm{m\cdot s^{-2}}$ when it moves horizontally. Find the vehicle's (uniform) acceleration on a 10% grade slope assuming that the same thrust is applied in both cases. The acceleration due to gravity is $g = 9.81\,\mathrm{m\cdot s^{-2}}$. *f(Aleš) transporting the competition prizes.*

The slope grade can be related to the angle of inclination α as

$$\mathrm{tg}\,\alpha = \frac{1}{10}\,.$$

The magnitude of acceleration a_s of the vehicle on the inclined surface can then be calculated as

$$a_\mathrm{s} = \frac{F_\mathrm{s}}{m}\,.$$

It remains to find an expression for the magnitude F_s of the force acting on the vehicle parallel to the inclined surface in terms of the magnitude F_v of the force which propelled the vehicle when it moved horizontally. We have

$$F_\mathrm{s} = F_\mathrm{v} - F_G = ma_\mathrm{v} - mg\sin\alpha\,,$$

because in the second case there is a non-zero component of gravitational force on the vehicle parallel to the inclined plane. Thus

$$a_\mathrm{s} = a_\mathrm{v} - g\sin\alpha\,.$$

Hence we get $a_\mathrm{s} \doteq 0.024\,\mathrm{m\cdot s^{-2}}$.

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Problem M.2 ... Náry's academic quarter

Most people's reactions can be virtually immediate but not in the case of Náry. Even when it comes to mere releasing of things to fall freely under gravity, it takes him $\tau = 6\,\mathrm{s}$ to react on a signal. Find the time (in seconds) which elapses until the distance between a body released by Náry and a body released immediately on signal grows to $l = 200\,\mathrm{m}$. The acceleration due to gravity is $g = 9.81\,\mathrm{m\cdot s^{-2}}$ and we start measuring time at the exact moment when the first body is released. *f(Aleš) lamenting Náry's late arrivals...*

Let us denote by t the time elapsed since we released the first body. Clearly, the time elapsed since Náry released his body is then $t - \tau$. The distances through which the bodies fall after time t can be expressed as

$$s(t) = \frac{1}{2}gt^2\,,$$
$$s_\mathrm{N}(t) = \frac{1}{2}g\,(t-\tau)^2\,\Theta(t-\tau)\,,$$

where Θ is the Heaviside step function which is defined to be zero at negative values and 1 elsewhere. The distance $l = s - s_{\mathrm{N}}$ happens to be greater than $s(\tau)$, so $t > \tau$ and

$$l = g\tau\left(t - \frac{1}{2}\tau\right) ,$$

thus

$$t = \frac{l}{g\tau} + \frac{\tau}{2} .$$

Hence we get $t \doteq 6.4\,\mathrm{s}$.

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Problem M.3 ... truly solid lecture notes

As we all know, some books do not really count as light-weights, so instead of carrying them in a bag let's try and drag them on the floor on a string. Consider a book lying on the floor and a string with length $l = 2.0\,\mathrm{m}$ whose one end is attached to the centre of the upper side of book's cover. The other end of the string is held in height $h = 1.2\,\mathrm{m}$ above the floor. Find the coefficient of friction between the book and the floor knowing that the book which is of mass $m = 1\,\mathrm{kg}$ is dragged at constant speed with force of magnitude $F = 5\,\mathrm{N}$. Assume that the book does not open. The acceleration due to gravity is $g = 9.81\,\mathrm{m{\cdot}s^{-2}}$.

f(Aleš)'s bag is getting heavier and heavier.

The coefficient of friction is can be calculated as a ratio of magnitudes of the force due to friction and the net force on the book normal to the floor. The magnitude F_{f} of the force due to friction must be equal to magnitude of the horizontal component of the force applied on the string, i.e.

$$F_x = F\cos\alpha = F\frac{\sqrt{l^2 - h^2}}{l} ,$$

where α is the angle between the string and the horizontal and the Pythagorean theorem was used to express $\cos\alpha$ in terms of l and h.

The force on the book normal to the floor is found to be the book's weight from which the vertical component of the force applied to the string was subtracted. Hence

$$F_{\mathrm{N}} = F_{\mathrm{G}} - F_y = mg - F\sin\alpha = mg - F\frac{h}{l} .$$

The coefficient of friction f is then

$$f = \frac{F_{\mathrm{f}}}{F_{\mathrm{N}}} = \frac{F\sqrt{l^2 - h^2}}{mgl - Fh} .$$

Hence we get $f \doteq 0.59$.

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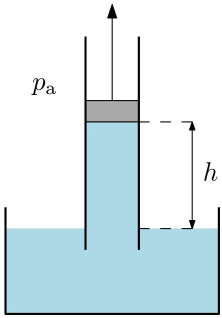
Problem M.4 ... all in vain

Consider a long pipe whose one end is plunged into water. Inside the pipe, there is a piston which perfectly fits between the walls. Having squeezed the air out, we start pulling the piston up, thus lifting a column of water in the pipe under the piston head. What is the maximum height h in which we can lift the water assuming that the ambient atmospheric pressure is $p_a = 760\,\mathrm{mmHg}$? *f(Aleš) was short of water.*



The water column under the piston head will continue to rise until the hydrostatic pressure will balance the ambient atmospheric pressure. Since we are given the ambient atmospheric pressure as the length of a mercury column, we can obtain the correct numerical answer by first multiplying its value by density of mercury $13\,595\,\mathrm{kg\cdot m^{-3}}$ and then dividing by density of water $1\,000\,\mathrm{kg\cdot m^{-3}}$. Hence the height at which the water column breaks is $h \doteq 10.3\,\mathrm{m}$.

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Problem E.1 ... the long one and the wide one

Consider two homogeneous cylindrical copper wires of equal mass, one of them being thrice as long as the other. Find the ratio of electrical resistances along the two wires, longer to the shorter. *Lukáš likes to steal lab equipment.*

Given resistivity of the material ϱ, length of the wire l and its cross-sectional area S, the electrical resistance along the wire can be found as

$$R = \frac{\varrho l}{S}\,.$$

The ratio of lengths of the two wires is 3. Since the two wires have the same mass, their volumes must also be the same and so the ratio of their cross-sectional areas must be 1/3. Since the cross-sectional area appears in the denominator of the expression for the resistance, we get the ratio to be 9. Hence the resistance of the longer wire is nine times as large as that of the shorter wire.

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Problem E.2 ... one, two, three,...

Náry was asked to find out the number of turns in his brand new copper coil. Too lazy to count them one by one, he resorted to his multimeter and good old data tables. Having brought a potential difference $U = 7.8\,\mathrm{V}$ across the coil, he measured a current $I = 2\,\mathrm{A}$ flowing in the coil. Also, he measured the cross-sectional diameter of the wire to be $S = 1\,\mathrm{mm^2}$. He then used his data tables to find the electrical resistance of copper to be $\varrho = 1.8\cdot 10^{-7}\,\Omega\cdot\mathrm{m}$. He also measured the diameter of the turns and obtained an average value $d = 6\,\mathrm{cm}$. Assuming that the total length of leads is $l_0 = 18\,\mathrm{cm}$, find the number of turns in Náry's coil.
f(Aleš) wanted to set a black-box problem.

We can use Ohm's law to find the resistance along the wire. This resistance is then directly proportional to the length of the wire, which may then be used to find the number of turns. We have

$$N = \frac{l - l_0}{2\pi \frac{d}{2}} = \frac{\frac{RS}{\varrho} - l_0}{\pi d} = \frac{\frac{US}{I\varrho} - l_0}{\pi d} \, .$$

Hence, we get $N \doteq 114$. It can be shown that the error we made by neglecting the helicity of the coil is on third decimal place.

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Problem E.3 ... experimental afternoon

The power outlets in a building where a FYKOS camp was held provide voltage $U_0 = 230\,\mathrm{V}$. However, some of the experiments had to be performed outside, especially those which involved shooting with an air gun. To this end, an extension lead of length $l = 101\,\mathrm{m}$ and resistance $R_0 = 1.4\,\Omega$ was used to bring the power supply where it was needed. Consider connecting a load to the extension lead. The properties of the load are such that if we bring a potential difference $U_s = U_0$ to it, the load provides power $P_s = 1\,000\,\mathrm{W}$ to an appliance (e.g. a heating spiral). Assuming that voltage $U_s = U_0$ is brought to the other end of the lead, find the potential drop across the load. *f(Aleš) had a moment of nostalgia.*

The whole circuit can essentially be thought of as consisting of two components: the load and the lead. The current flowing through both of these must be the same, so the Ohm's law yields

$$I = \frac{U_0}{R_0 + R_\mathrm{s}} \, ,$$

where R_s is the resistance of the load. Clearly, we must also have

$$I = \frac{U}{R_\mathrm{s}} \, ,$$

where U is the potential drop across the load and R_s is the resistance of the load. Equating the two expressions for I yields an equation for U. It remains to find an appropriate expression for R_s. Clearly, this is provided by knowing the power supplied by the load, when a potential difference U_0 is brought to it. We have

$$P_\mathrm{s} = \frac{U_0^2}{R_\mathrm{s}} \, .$$

Combining the two equations we derived, we obtain

$$U = \frac{U_0}{1 + \frac{R_0}{R_\mathrm{s}}} = \frac{U_0^3}{U_0^2 + R_0 P_\mathrm{s}} \, .$$

Plugging in the numbers, we get $U \doteq 224\,\mathrm{V}$.

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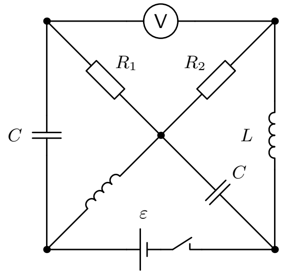
Problem E.4 ... pyramid's circuits

In the circuit shown below we first put the switch on and wait for the currents to come to a steady state. Find the magnitude of the potential difference (in volts) measured by the voltmeter. Take the capacitors and coils to be ideal and assume that the voltmeter provides an infinite resistance. Put $\varepsilon = 1\,\mathrm{V}$. The resistance $R_1 = 3R_2$.

Xellos found out about the Estonian-Finnish Olympiad in Physics.



We'll calculate currents through the resistors from the top vertex of the pyramid (junction in the center of the picture).

After the currents reach steady state, the capacitors and the voltmeter can be thought of as being elements with infinite resistance. On the other hand, the coil behaves like a conductor with no resistance. Hence the currents through R_1 and R_2 are $I_1 = 0$ and $I_2 = \varepsilon/R_2$ respectively, so the potential difference across the two resistors is

$$|R_1 I_1 - R_2 I_2| = \varepsilon\,.$$

The reading shown by the voltmeter is therefore $1\,\mathrm{V}$.

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Problem X.1 ... uncertain bacteria

Use a quantum uncertainty principle to find the uncertainty Δv_x in the component v_x of the velocity of a bacterium with mass $m = 1.0\cdot 10^{-15}\,\mathrm{kg}$ and linear size $l = 1.0\cdot 10^{-6}\,\mathrm{m}$. Assume that the position of the bacterium can be determined with uncertainty $\Delta x = l/100$ and that $v_x = 3.0\cdot 10^{-5}\,\mathrm{m\cdot s^{-1}}$. Submit your answer as $\log_{10}(\Delta v_x/v_x)$.

Mirek was trying to figure out why it is so difficult to determine the origin of some diseases.

The Heisenberg uncertainty principle relating the bacterium's position and momentum

$$\Delta x \Delta p_x = \frac{\hbar}{2}\,,$$

can obviously be recast as

$$\Delta x \Delta v_x = \frac{\hbar}{2m}\,.$$

Hence the ratio $\log_{10}(\Delta v_x/v_x)$ can be expressed as

$$\log_{10}\frac{\Delta v_x}{v_x} = \log_{10}\frac{\hbar}{2m\Delta x v_x} \doteq -7\,.$$

The order of magnutude of the uncertainty is -7.

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Problem X.2 ... a lecture on relativity

Two professors pass each other in a corridor at the Institute of Theoretical Physics, going in opposite directions. One is coming back from a lecture about special theory of relativity and his speed is $v_s = 1\,\mathrm{m\cdot s^{-1}}$, while the second one is going to a lecture about general relativity and his speed is $v_g = 2\,\mathrm{m\cdot s^{-1}}$. Determine the error that we will make by using classical physics instead of relativity to calculate the speed of the second professor in the reference frame of the first professor. *Mirek trying to entertain himself while waiting for a lecture on relativity.*

The classical result on addition of velocities says that

$$u = v_\mathrm{s} + v_\mathrm{g}\,,$$

whereas the corresponding relativistic expression can be written as

$$w = \frac{v_\mathrm{s} + v_\mathrm{g}}{1 + \frac{v_\mathrm{s} v_\mathrm{g}}{c^2}}\,.$$

The error made by using classical physics instead of relativity is therefore

$$u - w = (v_\mathrm{s} + v_\mathrm{g})\left(1 - \frac{1}{1 + \frac{v_\mathrm{s} v_\mathrm{g}}{c^2}}\right) = (v_\mathrm{s} + v_\mathrm{g})\frac{1}{\frac{c^2}{v_\mathrm{s} v_\mathrm{g}} + 1} \approx \frac{v_\mathrm{s} v_\mathrm{g}(v_\mathrm{s} + v_\mathrm{g})}{c^2} \doteq 6.7 \cdot 10^{-17}\,\mathrm{m\cdot s^{-1}}\,.$$

Error made by considering classical theory equals $6.7 \cdot 10^{-17}\,\mathrm{m\cdot s^{-1}}$.

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Problem X.3 ... hot grating

Determine the temperature needed to observe diffraction of a mono-energetic $^{20}\mathrm{Ne}$ beam (produced by an energy filter) on a grating with slit spacing $d = 0.5\,\mathrm{nm}$. The condition for diffraction to be observed is taken to be $\lambda = d$ where λ is the de Broglie wavelength of the atoms in the beam. *Mirek skimming through a quantum physics textbook.*

Kinetic energy of monoatomic particles is

$$E = \frac{3}{2}kT\,,$$

where $k = 1.38 \cdot 10^{-23}\,\mathrm{J\cdot K^{-1}}$ is the Boltzmann constant. By considering de Broglie relation

$$p = \frac{h}{\lambda}$$

we can write

$$E = \frac{p^2}{2m} = \frac{h^2}{2m\lambda^2}\,,$$

where m is the atomic mass of neon, which can be calculated as a product of atomic mass unit $a_\mathrm{u} = 1.67 \cdot 10^{-27}\,\mathrm{kg}$ and the relative atomic mass of neon $A_\mathrm{Ne} = 20$. Hence the temperature is given as

$$T = \frac{h^2}{3mkd^2} \doteq 1.3\,\mathrm{K}\,.$$

The needed temperature to see the difraction is $T \doteq 1.3\,\mathrm{K}$.

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Problem X.4 ... beta decay

Consider a beam of neutrons, where every neutron in the beam has kinetic energy $E_k = 0.05\,\mathrm{eV}$. Take the half-life and the rest energy of a neutron to be $T_{1/2} = 640\,\mathrm{s}$ and $m_n c^2 = 940\,\mathrm{MeV}$, respectively. Find the fraction of the number of neutrons which will decay before the beam travels a distance of $d = 10\,\mathrm{m}$. *Mirek got inspired by a textbook about particle physics.*

It is obvious that $m_\mathrm{n} c^2 \gg E_\mathrm{k}$, therefore we can treat this problem classically. In such a case, speed of a particle can be written as

$$v = \sqrt{\frac{2E_\mathrm{k}}{m_\mathrm{n}}}$$

and so the time in which it travels distance d is

$$t = \frac{d}{v} = \frac{d}{c}\sqrt{\frac{m_\mathrm{n}c^2}{2E_\mathrm{k}}}\,.$$

It remains to apply the law of radioactive decay in the form

$$N(t) = N_0 \mathrm{e}^{-(t\ln 2)/T_{1/2}}$$

using which we find the fraction of neutrons which will decay, i.e.

$$\frac{N_0 - N}{N_0} = 1 - \mathrm{e}^{-(t\ln 2)/T_{1/2}} = 1 - \mathrm{e}^{-\frac{d\ln 2}{T_{1/2}c}\sqrt{\frac{m_\mathrm{n}c^2}{2E_\mathrm{k}}}} \approx \frac{d\ln 2}{T_{1/2}c}\sqrt{\frac{m_\mathrm{n}c^2}{2E_\mathrm{k}}} \doteq 3.5\cdot 10^{-6}\,.$$

Only $3.5 \cdot 10^{-6}$ of the neutrons will decay.

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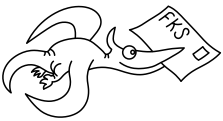

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FYKOS is organized by students of Faculty of Mathematics and Physics of Charles University. It's part of Public Relations Office activities and is supported by Institute of Theoretical Physics, MFF UK, his employees and The Union of Czech Mathematicians and Physicists.