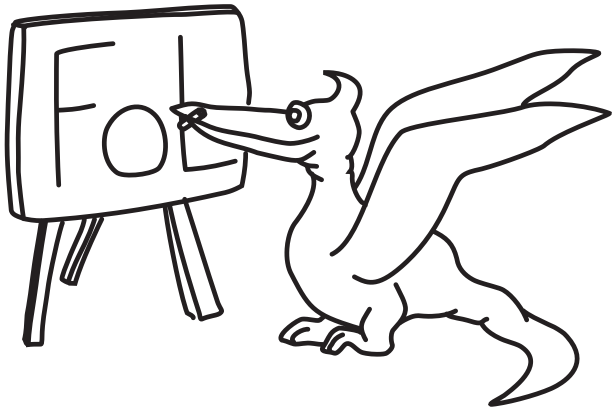
Solutions of 6th Online Physics Brawl



Problem FoL.1 ... librarian's problem

A stack of 40 books lies on the ground. Each of them is $d = 2\,\mathrm{cm}$ *thick and has mass* $m = 1\,\mathrm{kg}$. *We want to place the books into 4 shelves, with each shelf containing a stack of 10 books lying on the side and the shelves' are at heights* $100\,\mathrm{cm}$, $130\,\mathrm{cm}$, $160\,\mathrm{cm}$ *a* $190\,\mathrm{cm}$. *What work do we have to perform to move the books?* *Mirek was unable to move full bookshelves.*

This problem is simple, we just need to pick the right approach and avoid dealing with each book individually, but only with the center of mass of all books before and after the process. Initially, the center of mass of the books is at the height $h = 40\,\mathrm{cm}$ (in the middle of the homogeneous stack); after they are moved to the shelves, the center of mass is at a height

$$h' = \frac{1}{4}\Big(100\,\mathrm{cm} + 5d + 130\,\mathrm{cm} + 5d + 160\,\mathrm{cm} + 5d + 190\,\mathrm{cm} + 5d\Big) = 155\,\mathrm{cm}\,.$$

The work is then given as the change in potential energy

$$W = \Delta E_\mathrm{p} = 40mg(h' - h) \doteq 450\,\mathrm{J}\,.$$

In order to place the books on the shelves, we need to perform work $W = 450\,\mathrm{J}$.

Miroslav Hanzelka
`mirek@fykos.cz`

Problem FoL.2 ... tough slice of bread

A square wheat field covers an area of $16\,\mathrm{ha}$. *A harvesting company owns a combine harvester whose reel is* $8\,\mathrm{m}$ *wide. The combine moves with velocity* $5\,\mathrm{km{\cdot}h^{-1}}$ *and it consumes 10 litres of fuel per kilometer when harvesting. The company buys the fuel for* 28.40 *CZK per litre. How much will harvesting the whole field cost the company (rounded to integer CZK), if they have to pay the combine driver* 85 CZK *for each started hour of work? Neglect any delays of the combine due to turning or fires in the field.* *Meggy was harvesting part-time.*

One side of the field has length $400\,\mathrm{m}$. We have $400/8 = 50$, so the combine has to cross the whole field along one side 50 times. That way, the length of its path will be at least $20{,}000\,\mathrm{m}$, it will consume $200\,\mathrm{l}$ of fuel and take $4\,\mathrm{h}$ to do so. The company has to pay 5,680 CZK for the fuel and 340 CZK to the driver. The total cost of harvesting is 6,020 CZK.

Markéta Calábková
`calabkovam@fykos.cz`

Problem FoL.3 ... curve

Olda is driving a car with velocity $v = 20\,\mathrm{m{\cdot}s^{-1}}$*; he keeps an air freshener on his rear view mirror. When crossing a curve, his air frshener tilted by* $\alpha = 35^\circ$. *What's the radius of the curve Olda was crossing? The acceleration due to gravity is* $g = 9.81\,\mathrm{m{\cdot}s^{-2}}$.
Olda was driving through a curve.

First of all, we need to realise that gravity is perpendicular to the (flat) road. In addition, there's the centrifugal force given by

$$F_\mathrm{c} = ma_\mathrm{c} = \frac{mv^2}{r}\,.$$

The force of gravity is given by

$$F_\mathrm{G} = mg\,.$$

Simple reasoning leads us to the conclusion that

$$\operatorname{tg}\alpha = \frac{F_\mathrm{c}}{F_\mathrm{G}}\,,$$
$$mg\operatorname{tg}\alpha = \frac{mv^2}{r}\,,$$
$$r = \frac{v^2}{g\operatorname{tg}\alpha}\,.$$

The radius of the curve is $r = 58.23\,\mathrm{m}$.

Oldřich Holcner
holcner@fykos.cz

Problem FoL.4 ... astral

Dr Strange is an impenitent pragmatist who does not believe in out-of-body experiences. A nun wants to correct his mistake by hitting him in the chest with a constant force and knocking the astral body (with mass identical to that of Dr Strange, $m = 80\,\mathrm{kg}$) out of him. If the impact (contact of the fist and the chest) took $t = 0.1\,\mathrm{s}$ and the astral body was moving with velocity $v = 5\,\mathrm{m{\cdot}s^{-1}}$ afterwards, what force did the nun hit with? The body of Dr Strange itself does not move, violation of mass conservation is not our problem.

Mirek was relaxing and watching trailers.

This is a simple problem on the topic of impulse

$$F = \frac{p}{t}\,.$$

The force we are looking for is therefore

$$F = \frac{mv}{t} = 4,000\,\mathrm{N}\,.$$

The nun hits Dr Strange with force $F = 4\,\mathrm{kN}$.

Miroslav Hanzelka
mirek@fykos.cz

Problem FoL.5 ... don't mind the tune

Meggy brought a flute to a campfire. The flute is made of the ABS polymer with thermal expansion coefficient $\alpha = 9\cdot 10^{-5}\,\mathrm{K^{-1}}$. When she tried playing it indoors at $20\,^\circ\mathrm{C}$, its lowest tone had frequency $349\,\mathrm{Hz}$ and the flute had length l. Near the fire, she placed the flute on a bench. Assuming that the period of sound waves forming each tone is proportional to the length of the flute, determine the frequency of its lowest tone, if the flute got heated up to $25\,^\circ\mathrm{C}$ in the meantime. *Meggy was creating problems during a Fykos camp.*

The elongation of the flute is computed easily, $\Delta l = l\alpha\Delta t$. After substituting ($\alpha$ in $\mathrm{K^{-1}}$ and Δt in K), we get $\Delta l = 4.5\cdot 10^{-4} l$. Let us denote the frequency of the tune at $20\,^\circ\mathrm{C}$ by f_1 and the frequency at 25 °C by f_2. Since a period is just the inverse value of frequency, we obtain a formula for the new frequency: $(1 + 4.5 \cdot 10^{-4}) f_2 = f_1$, so $f_2 = 348.843\,\mathrm{Hz}$.

Markéta Calábková
calabkovam@fykos.cz

Problem FoL.6 ... little cubes

Assume the molecules of water to be cubes in close contact (the volume of water is completely filled by the cubes). Determine the number of molecules that fit in $V' = 1\,\mathrm{nm}^3$ of volume. Consider the water to be at standard conditions ($T \approx 300\,\mathrm{K}$, $p \approx 1\,\mathrm{atm}$), look up the necessary values of water density, molar mass and Avogadro's number (if you don't know them by heart).

Mirek didn't see the point of including such primitive problems in master's studies.

We know the density of water ϱ, its molar mass M and Avogadro's number N_A. The mass of one molecule is $m = M/N_\mathrm{A}$, its volume is $V = a^3$, so

$$a = \sqrt[3]{\frac{m}{\varrho}} = \sqrt[3]{\frac{M}{N_\mathrm{A}\varrho}} \doteq 3.1 \cdot 10^{-10}\,\mathrm{m}\,.$$

Therefore, the number of molecules in $V' = 1\,\mathrm{nm}^3$ is

$$N = \frac{V'}{a^3} \doteq 33\,.$$

Here, we used well-known approximate values $\varrho = 1\,000\,\mathrm{kg{\cdot}m^{-3}}$, $N_\mathrm{A} = 6.022 \cdot 10^{23}\,\mathrm{mol^{-1}}$ and $M = 18\,\mathrm{g{\cdot}mol^{-1}}$.

Miroslav Hanzelka
mirek@fykos.cz

Problem FoL.7 ... methane

At the bottom of, for example, the South China Sea, is a substance know as methane clathrate. It's a crystal of water ice with gaps in which methane molecules are enclosed. These gaps are present in common ice as well, but they aren't filled by any substance. The chemical formula of this clathrate is $CH_4 \cdot 5.75H_2O$. How many m^3 of methane can be released to the atmosphere at temperature $0\,^\circ\mathrm{C}$ and pressure $100\,\mathrm{kPa}$ from $1\,\mathrm{m}^3$ of the clathrate? Water ice density is $917\,\mathrm{kg{\cdot}m^{-3}}$.

Sweet memories of Katka's high school years.

Since we're dealing with water ice with density of $917\,\mathrm{kg{\cdot}m^{-3}}$, one cubic metre contains 917 kg of ice. Since one mole of water weighs 18 g, the amount of ice in one cubic metre corresponds to 50.9 kmol. There's 5.75 times less methane, which is 8.86 kmol. To determine the volume of methane, we can use the ideal gas law

$$V = \frac{nRT}{p}\,.$$

Numerically, $V \doteq 201\,\mathrm{m}^3$.

Kateřina Smítalová
katka@fykos.cz

Problem FoL.8 ... do it tomorrow

Compute how long would day on Earth have to be in order to cause weightlessness at the equator. The Earth's radius is $R = 6,400\,\mathrm{km}$*, its mass is* $M = 6 \cdot 10^{24}\,\mathrm{kg}$*, the gravitational constant is* $G = 6.7 \cdot 10^{-11}\,\mathrm{kg^{-1}{\cdot}m^3{\cdot}s^{-2}}$*. Assume the Earth to be spherical.*

Mirek was sitting by the window watching time fly.

The gravitational acceleration at the equator is

$$a_\mathrm{g} = \frac{GM}{R^2}$$

and it has to be balanced by centrifugal acceleration at the equator

$$a_\mathrm{c} = \omega^2 R$$

in order to yield zero net acceleration. Thus, we get

$$\omega = \frac{2\pi}{T} = \sqrt{\frac{GM}{R^3}}\,,$$

from which

$$T = 2\pi\sqrt{\frac{R^3}{GM}} \doteq 5,100\,\mathrm{s} \doteq 1.4\,\mathrm{h}\,.$$

The day would have to last 1.4 h.

Miroslav Hanzelka
mirek@fykos.cz

Problem FoL.9 ... Toricelli reloaded

We are on a space station with standard atmospheric pressure $p_\mathrm{a} = 101\,\mathrm{kPa}$ *inside, performing the Toricelli experiment. We take a bottle of mercury, stick a long test tube inside and when it is full, place it vertically in such a way that the open end is submerged under the surface of mercury in the bottle. During the experiment, the space station is sufficiently far from the gravitational influence of any celestial bodies and is accelerating in the direction of the closed end of the test tube (after the above described manipulation) with acceleration* $a = 20\,\mathrm{m{\cdot}s^{-2}}$*. How high will the mercury column in the test tube be? The density of mercury is* $\varrho = 13\,600\,\mathrm{kg{\cdot}m^{-3}}$*.*

Mirek was disappointed by an easy lecture, so he upgraded the problems.

The hydrostatic pressure in the test tube after the experiment ends has to balance the surrounding pressure, which is $p_\mathrm{a} \doteq 101\,\mathrm{kPa}$. We can compute the hydrostatic pressure using the formula

$$p = h\varrho a\,,$$

replacing the usual acceleration due to gravity g by the acceleration of the space station a. From the formula

$$p_\mathrm{a} = p = h\varrho a$$

we express

$$h = \frac{p_\mathrm{a}}{\varrho a} \doteq 0.37\,\mathrm{m}\,.$$

The column of mercury is approximately half as high as would be on Earth, since the acceleration of the space station is approx. twice as large.

Miroslav Hanzelka
mirek@fykos.cz

Problem FoL.10 ... effectiveness evaluation

Karel and Aleš have one homogeneous brick each. The bricks have cuboid shapes with edge lengths $a \times a \times 2a$, where $a = 5\,\mathrm{cm}$. The mass of each brick is $m = 1\,\mathrm{kg}$. Karel puts his brick down and starts flipping it in one direction always about the shorter edge (looking from the side, the longer edge and the shorter edge alternate in touching the ground), until the brick crosses a distance $d = 150\,\mathrm{cm}$. Aleš is also flipping his brick until it crosses the distance d, but he is flipping it about the longer edges only (only the square base of the cuboid is ever visible from the side). The bricks do not slip and collisions with the ground are inelastic. How much more work will Karel perform? The acceleration due to gravity is $g = 9.81\,\mathrm{m\cdot s^{-2}}$.

Mirek was evaluating the others' performances.

For Karel to move the brick from lying down state to standing state, he has to move the center of mass from the height $a/2$ to the height $a\sqrt{5}/2$, with the brick standing on an edge and the center of mass is the highest. The rest of the rotation is managed by gravity. Then, he has to lift it again from a to $a\sqrt{5}/2$, so that it would be able to fall back to its lying down state. During these actions, the brick moves by $3a = 15\,\mathrm{cm}$, so both flips have to be performed 10 times. The total energy that Karel used, and thus the work done is

$$W_\mathrm{K} = 10\left(a\left(\frac{\sqrt{5}}{2} - \frac{1}{2}\right) + a\left(\frac{\sqrt{5}}{2} - 1\right)\right) mg = 10\left(\sqrt{5} - \frac{3}{2}\right) mga\,.$$

Aleš has to flip the brick 30 times about an edge, moving the center of mass from $a/2$ to $a\sqrt{2}/2$, so his total work is

$$W_\mathrm{A} = 30\left(\frac{\sqrt{2}}{2} - \frac{1}{2}\right) mga\,.$$

The difference in work performed by Karel and Aleš is

$$W_\mathrm{K} - W_\mathrm{A} = \left(10\sqrt{5} - 15\sqrt{2}\right) mga \doteq 0.56\,\mathrm{J}\,.$$

Karel performed 0.56 J more work than Aleš.

Miroslav Hanzelka
mirek@fykos.cz

Problem FoL.11 ... 'murican

Náry does not like unpleasant surprises. How far from the muzzle of an air gun does he have to stand in order to hear the shot exactly one second before he would get hit? The speed of sound is $334\,\mathrm{m\cdot s^{-1}}$, the exit velocity of the bullet is $152\,\mathrm{m\cdot s^{-1}}$, the elevation angle of the shot is $10\circ$. Neglect air resistance. The term "heard" is considered equivalent to Náry being reached by the sound wave. The time the bullet spends inside the muzzle is also negligible. Compute the answer to metre precision.

Kiki would shoot.

The x-component of the path crossed by the bullet will be the same as the length of sound's path. Knowing the time delay of the shot, we can write this fact as

$$v_0 t \cos\alpha = v(t-1)\,,$$

where v_0 is the initial velocity of the shot, t is the time the shot takes to hit Náry, α is the elevation angle and v is the speed of sound. The time t can be expressed as

$$t = \frac{v}{v - v_0\cos\alpha}$$

and substituted into the formula for the x-component of the bullet's displacement

$$x = v_0 \frac{v}{v - v_0\cos\alpha}\cos\alpha\,.$$

After plugging in the numerical values, we get $x \doteq 271\,\mathrm{m}$. Náry would theoretically have to stand approximately 271 m away from the air gun's muzzle.

Kristína Nešporová
kiki@fykos.cz

Problem FoL.12 ... from Prague to Brno

You are driving from Prague to Brno with velocity $v = 150\,\mathrm{km\cdot h^{-1}}$ on the D1 motorway. The length of this path is $d = 196\,\mathrm{km}$. How many times will you pass by a RegioJet bus (also heading to Brno) during driving, if you know that every half hour, a bus leaves Prague for Brno with velocity $v_\mathrm{b} = 80\,\mathrm{km\cdot h^{-1}}$ and you leave Prague at the same time as one bus? (Don't count that bus, However, if you meet a bus when arriving at Brno, do count that one.) Assume that there are no traffic jams or closed sections of the motorway due to construction.

Dominika had an idea when travelling back from a FYKOS camp.

Let us say that you pass by the first bus at a distance s from Prague. When you left Prague, the bus was already

$$s(t_0) = v_\mathrm{b} t_0$$

($t_0 = 0.5\,\mathrm{h}$) away from you. Until you meet, the bus crosses an additional distance s_b. It's clear that

$$s(t_0) + s_\mathrm{b} = s\,.$$

The time you took to cross the distance s and the bus took to cross the distance s_b must be identical:

$$\frac{s_\mathrm{b}}{v_\mathrm{b}} = \frac{s}{v}\,.$$

From the above, the distance s can be computed easily as

$$s = \frac{v_\mathrm{b} v}{v - v_\mathrm{b}} t_0\,.$$

As soon as you pass by that bus, your situation is the same with respect to the next bus in front of you, so it is enough to divide the length of the motorway by the distance s and we get the number of buses you meet:

$$\left\lfloor \frac{d(v - v_\mathrm{b})}{v_\mathrm{b} v t_0} \right\rfloor\,.$$

After plugging in the given values, the result is 2.

Dominika Kalasová
dominika@fykos.cz

Problem FoL.13 ... and the Sun in Ophiuchus means that...

What is the maximum force that Jupiter can exert on a FYKOS member on Earth? The FYKOS member has mass $m_\mathrm{F} = 85.6\,\mathrm{kg}$, look up the remaining parameters. Assume that the orbits of Jupiter and Earth are circular with radii equal to the lengths of their semi-major axes and coplanar. *Karel heard that this is good argument for weak believers in astrology.*

Let us look up the lengths of semi-major axes of Earth's orbit $a_\mathrm{Z} = 1.50 \cdot 10^{11}\,\mathrm{m}$ and Jupiter's orbit $a_\mathrm{J} = 7.78 \cdot 10^{11}\,\mathrm{m}$. The planets will be farthest or closest from each other exactly when they are collinear with the Sun. Since we are interested in the maximum force between the FYKOS member and Jupiter, we consider the minimum distance between the planets. Relative to that distance, the exact position of the FYKOS member is negligible and the minimum distance FYKOS member – Jupiter is simply $r_\mathrm{min} = a_\mathrm{J} - a_\mathrm{Z}$. The force between them is

$$F_\mathrm{g} = G\frac{m_\mathrm{F} m_\mathrm{J}}{r_\mathrm{min}^2}\,,$$

where $G = 6.67 \cdot 10^{-11}\,\mathrm{N{\cdot}kg^{-2}{\cdot}m^2}$ is the gravitational constant and $m_\mathrm{J} = 1.90 \cdot 10^{27}\,\mathrm{kg}$ is the mass of Jupiter. When we plug it into the formula, we get the maximum force $F \doteq 2.75 \cdot 10^{-5}\,\mathrm{N}$. That's really very little.

Karel Kolář
karel@fykos.cz

Problem FoL.14 ... bounce from zero

In the xy plane, there is a perfectly reflecting mirror represented by the curve $x^2 + y^2 = 1$, $y < 0$. We send a light ray towards the concave side of the mirror along the line $x = \sqrt{3}/2$. How many times will the ray reflect before flying back to the half-plane $y > 0$?

Mirek was too lazy to turn.

The angle between the ray and the tangent to the half-circle at the incidence point is simply

$$\alpha = \arccos\left(\frac{\sqrt{3}}{2}\right) = 30^\circ\,,$$

since the circle has unit radius. Next, we know that the incidence angle is equal to the reflection angle (here, we work with the 90° complement) and that the central angle of a half-circle is 180°. Since the incident ray is parallel to the y axis, the central angle corresponding to the segment connecting two successive reflection points is equal to 2α. For the number of reflections n, we can write an inequality

$$\alpha + 2(n-1)\alpha < 180^\circ\,,$$

where the left hand side represents the angle by which the ray has turned (or equivalently the angle, "at which we see" the n-th reflection point from the center of the half-circle). We can look for the largest integer n satisfying the inequality. After expressing n,

$$n < \frac{1}{2}\left(\frac{180^\circ}{\alpha} + 1\right) = \frac{7}{2}\,,$$

so the sought for number of reflections is $n = 3$. This result can be seen almost instantly if we realise that the segments connecting successive reflection points are just sides of a regular hexagon, so the ray turns around after three reflections.

Miroslav Hanzelka
`mirek@fykos.cz`

Problem FoL.15 ... St. Elmo's sphere

What's the minimum radius r of a conducting sphere such that there's no spontaneous discharge, if the charge of the sphere is $Q = 1\,\mathrm{C}$? In air, spontaneous discharge occurs if the electric field reaches $E_{\mathrm{max}} = 25\,\mathrm{kV{\cdot}cm^{-1}}$. *Erik likes easy problems.*

The electric field outside a charged ball is the same as for a point charge. Therefore, the el. field at the surface of the sphere is

$$E = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2} < E_{\mathrm{max}}\,.$$

From that inequality, we can express the radius

$$r > \sqrt{\frac{1}{4\pi\varepsilon_0}\frac{Q}{E_{\mathrm{max}}}} \doteq 60.0\,\mathrm{m}\,.$$

The radius of the sphere has to be at least $60\,\mathrm{m}$.

Erik Hendrych
`erik@fykos.cz`

Problem FoL.16 ... under pressure

We put a metal rod of length $l = 1\,\mathrm{m}$ into a hydraulic press. The ends of the rod are held in a way that creates a small stress σ_0. The rod's thermal expansion coefficient is $\alpha = 1.1{\cdot}10^{-5}\,\mathrm{K^{-1}}$ and its Young modulus is $E = 200\,\mathrm{GPa}$. We heat up the rod by $\Delta T = 1\,\mathrm{K}$. Calculate the change in stress in the rod if the rams of the press do not move and are not thermally conducting. *Mirek was watching a hydraulic press.*

If the rod was not held in the press, heating it would cause elongation according to the formula

$$\varepsilon = \frac{\Delta l}{l} = \alpha\Delta T\,,$$

where ε is the strain. Since the rod cannot extend in the press, it has to be compressed by this ε. The initial stress is irrelevant as long as we are still in the linear range of dependence of strain on stress (which is implied by the initial stress σ_0 being small). Hooke's law says that the stress increases by

$$\Delta\sigma = \varepsilon E = \alpha\Delta T E \doteq 2.2\cdot 10^6\,\mathrm{Pa}\,.$$

The stress in the rod increases by $\Delta\sigma = 2.2\,\mathrm{MPa}$.

Miroslav Hanzelka
`mirek@fykos.cz`

Problem FoL.17 ... long drop

We throw a polystyrene ball of radius $r = 1.0\,\mathrm{cm}$ and density $\varrho = 0.06\,\mathrm{g\cdot cm^{-3}}$ vertically down into a ventillation shaft. In the shaft, hot warm air with density $\varrho_\mathrm{a} = 1.1\,\mathrm{kg\cdot m^{-3}}$ flows up with velocity $v_0 = 10\,\mathrm{m\cdot s^{-1}}$. Determine the magnitude of terminal velocity of the ball with respect to the shaft, if the drag coefficient of the ball is $C = 0.50$ and the acceleration due to gravity is $g = 9.8\,\mathrm{m\cdot s^{-2}}$. *Mirek couldn't work during backup, so he was creating problems.*

Gravity and air drag have to balance at terminal velocity,

$$mg = \frac{1}{2} C \varrho_\mathrm{a} S v^2\,;$$

for a ball with mass $m = (4/3)\varrho S r$, we can determine the terminal velocity with respect to static air

$$v = \sqrt{\frac{8gr\varrho}{3C\varrho_\mathrm{a}}} \doteq 5.3\,\mathrm{m\cdot s^{-1}}\,.$$

This is the terminal velocity of the ball with respect to air, so it will move up with velocity $v' = |v - v_0| = 4.7\,\mathrm{m\cdot s^{-1}}$ with respect to the shaft until it exits the shaft.

Miroslav Hanzelka
`mirek@fykos.cz`

Problem FoL.18 ... microrotation

The molar mass of water is $M = 18\,\mathrm{g\cdot mol^{-1}}$. Its molecules have a certain rotational energy with characteristic scale $\varepsilon = 650\,\mathrm{J\cdot mol^{-1}}$. The linear dimension of one molecule (its diameter) is $l = 310\,\mathrm{pm}$. Using dimensional analysis, determine the characteristic scale for the period of rotation. The answer is the base-10 logarithm of this period in seconds.
Mirek didn't see the point of including such primitive problems in master's studies.

From the given energy and length scales and the mass of one molecule, dimensional analysis lets us construct a formula

$$t = l\sqrt{\frac{M}{\varepsilon}}\,.$$

Simply plugging in the given values after conversion to basic SI units, we get

$$t \doteq 1.6 \cdot 10^{-12}\,\mathrm{s} = 1.6\,\mathrm{ps}\,.$$

The characteristic time scale of rotation of a water molecule is in the order of picoseconds.

Miroslav Hanzelka
`mirek@fykos.cz`

Problem FoL.19 ... perfectly balanced

Consider a thin homogeneous disk of radius r, supported in the middle. Initially, there were two point masses m placed on the perimeter of the disk opposite each other so that the system remained balanced. We want to replace one of the masses by a pair of masses $3m/5$ and $4m/5$, both placed on the perimeter of the disk, so that the system still remained balanced. How far *(in multiples of r) from each other do the two replacement masses have to be?*

Meggy was bored during an algebra lecture.

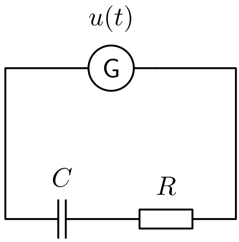
A mass $3m/5$ placed at the perimeter of the disk has the same effect as a mass m placed at a distance $3r/5$ from the middle. Let's denote the vector from the middle to this mass by $\boldsymbol{r}_1$. Similarly, a mass $4m/5$ at the perimeter has the same effect as a mass m at 4/5-ths of the radius; let's denote the vector to it by $\boldsymbol{r}_2$. These two masses together have the same effect as one mass m placed at the sum of vectors $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$. Therefore, these two vectors have to sum up to one vector of length r. We've got a triangle with sides of length $3r/5$, $4r/5$ and r, which is right-angled according to the Pythagorean theorem. The angle between vectors $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ is a right angle and the masses were supposed to be placed at the perimeter of the disk, so their distance is, again by the Pythagorean theorem, $\sqrt{2}\,r$.

Markéta Calábková
calabkovam@fykos.cz

Problem FoL.20 ... sine

We connect a tone generator producing a sine wave voltage $u(t)$ into a series RC circuit. The resistor in the circuit has resistance $R = 200\,\Omega$, the capacitor has capacity $C = 50\,\mu\mathrm{F}$, the maximum voltage of the source is $U = 5\,\mathrm{V}$. What's the mean charge on the capacitor?

Kuba simplified his first circuit.



The generator can be replaced by a DC source of voltage

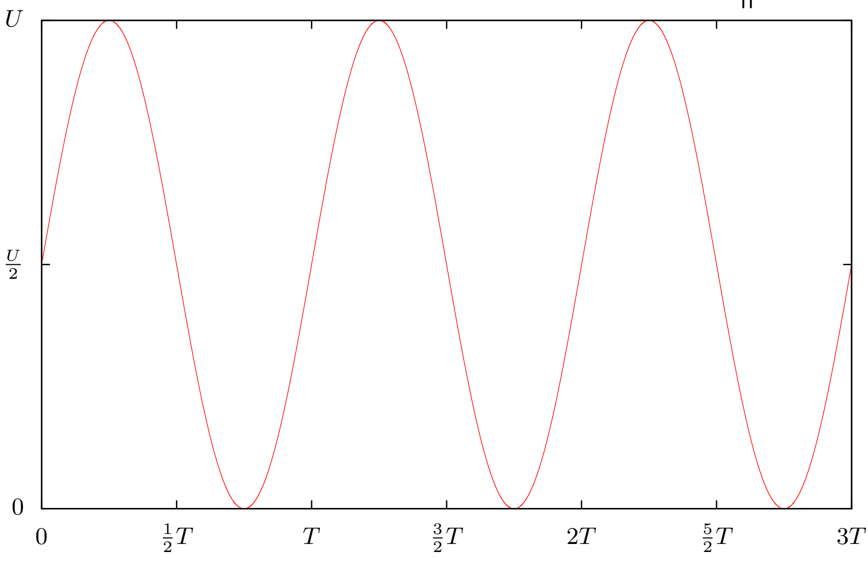


Fig. 1: Sine wave voltage.

$$U_1 = U/2$$

and an AC source of voltage

$$U_2(t) = \frac{U}{2} \sin\left(\frac{2\pi t}{T}\right)$$

connected in series.

All elements of the circuit are linear, so the current can be viewed as a superposition of two circuits, where each of them contains just one of these sources. In the DC circuit, the charge on the capacitor is constant

$$Q_1(t) = CU_1 = \frac{CU}{2}\,.$$

In the AC circuit, the charge on the capacitor harmonically changes (with some phase shift with respect to the voltage U_2) and its mean value is zero.

The total mean charge is a sum of mean charges in both circuits. Therefore, we get

$$\langle Q \rangle = Q_1 = \frac{CU}{2} = 125\,\mu\mathrm{C}\,.$$

The mean charge on the capacitor is $125\,\mu\mathrm{C}$.

Jakub Dolejší
krasnykuba@fykos.cz

Problem FoL.21 ... chain reaction

We placed a large number of balls of radius $r = 1\,\mathrm{cm}$ in a line. The distance of centers of any two successive balls is $d = 5\,\mathrm{cm}$. Each ball has mass $m = 20\,\mathrm{g}$ and rolling friction coefficient $\xi = 0.6\,\mathrm{mm}$. The first ball in the line is given a velocity $v = 1\,\mathrm{m\cdot s^{-1}}$ in order to start a series of central collisions of the balls. Which ball will be the last one to move? All collisions are perfectly elastic and the balls don't slip. The acceleration due to gravity is $g = 9.81\,\mathrm{m\cdot s^{-2}}$.

Mirek was strolling through the St. Wenceslas square.

Since the collisions are perfectly elastic, they occur without energy losses. The balls have equal masses, which means that when the first ball collides with the second one, the first one stops and the second one starts moving with velocity equal to the velocity of the first ball before the collision. The balls are completely identical, so we can view each collision as an instantaneous jump (displacement) of the incident ball by $2r$. Total kinetic energy of the first ball is

$$E_\mathrm{k} = \frac{1}{2}mv^2 + \frac{1}{2}J\omega^2\,.$$

Using $\omega = v/r$, $J = 2mr^2/5$ we get

$$E_\mathrm{k} = \frac{7}{10}mv^2\,.$$

The ball stops when the work done by the floor on the ball becomes equal to the initial kinetic energy. Since the rolling drag force is constant, we easily obtain

$$mg\frac{\xi}{r}s = \frac{7}{10}mv^2$$

and

$$s = \frac{7v^2 r}{10\xi g}\,,$$

where s is the distance covered by the first ball, neglecting other balls. There is one collision for every $d - 2r$ of length, thus the total number of collisions will be

$$\left\lfloor \frac{7v^2 r}{10(d-2r)\xi g} \right\rfloor = 39\,.$$

Since there are 39 collisions, the last ball to move is ball number 40.

Miroslav Hanzelka
mirek@fykos.cz

Problem FoL.22 ... ready steady

We placed two identically charged particles with masses $m = 1\,\mathrm{g}$ and charges $q = 1\,\mathrm{\mu C}$ at a height $h = 2\,\mathrm{m}$ above the ground. Afterwards, we release them and measure the velocity of one of the particles just before colliding with the ground to be $v = 10\,\mathrm{m\cdot s^{-1}}$. How much did the electrostatic potential energy decrease between the start of the movement and this moment? The acceleration due to gravity is $g = 9.81\,\mathrm{m\cdot s^{-2}}$.

Mirek was not so fondly remembering children's games.

Both particles are acted on by the vertical gravity and horizontal electrostatic force – it's horizontal because the particles are identical, so neither one can fall faster than the other. In order to find the horizontal velocity of a particle before impact, we need to determine the vertical component of the velocity. That's well-known:

$$v_\perp = \sqrt{2hg}\,,$$

so the horizontal component is

$$v_\parallel = \sqrt{v^2 - v_\perp^2}$$

and it appears in the law of energy conservation for the system in the form

$$\Delta E_\mathrm{e} = \Delta E_{\mathrm{k}\parallel} = 2 \cdot \frac{1}{2} m v_\parallel^2 = m\left(v^2 - 2hg\right);$$

This formula can be seen based on the law of energy conservation for vertical components. Numerically, the electrostatic potential energy change is $\Delta E_\mathrm{e} = 0.061\,\mathrm{J}$.

Miroslav Hanzelka
mirek@fykos.cz

Problem FoL.23 ... it hangs – it hangs

In a laboratory, a polystyrene ball of mass $m_1 = 1\,\mathrm{kg}$ and density $\varrho = 1\,020\,\mathrm{kg\cdot m^{-3}}$ hangs on a massless rope. The molar heat capacity of polystyrene is $c = 1\,400\,\mathrm{J\cdot kg^{-1}\cdot K^{-1}}$, its thermal expansion coefficient is $\alpha = 7 \cdot 10^{-5}\,\mathrm{K^{-1}}$. On another rope, we hang another ball made of the same material and with mass $m_2 = 0.5\,\mathrm{kg}$. To each ball, we transfer heat $Q = 5\,\mathrm{kJ}$, which increases the energy of the first ball by E_1 and of the second ball by E_2. Find the difference

$E_2 - E_1$. The acceleration due to gravity is $g = 9.81\,\mathrm{m{\cdot}s^{-2}}$.

Mirek was pondering textbook problems.

The balls are located in the gravity of Earth. After we add heat Q to a ball, its temperature increases by $\Delta T = Q/(mc)$ and its radius increases by

$$\Delta r = r_0 \alpha \Delta T\,.$$

The original radius can be determined using the mass and density to be

$$r_0 = \sqrt[3]{\frac{3m}{4\pi\varrho}}\,.$$

The change in radius also tells us how much the center of mass of a ball changed (it was originally at distance r_0 from one end of the respective rope, now it's at distance $r = r_0 + \Delta r$). The potential energy of a ball decreased by

$$mg\Delta r = mg\alpha\Delta T\sqrt[3]{\frac{3m}{4\pi\varrho}} = \frac{\alpha Q g}{c}\sqrt[3]{\frac{3m}{4\pi\varrho}}\,.$$

Since the same heat was transferred to each ball, the difference of energy changes is

$$E_2 - E_1 = m_2 g\Delta r_2 - m_1 g \Delta r_1 = \frac{\alpha Q g}{c}\left(\sqrt[3]{\frac{3m_2}{4\pi\varrho}} - \sqrt[3]{\frac{3m_1}{4\pi\varrho}}\right) \doteq 3.1\cdot 10^{-5}\,\mathrm{J}\,.$$

Let's note that if the balls were standing on a thermally insulating surface, the result would have the opposite sign.

Miroslav Hanzelka
mirek@fykos.cz

Problem FoL.24 ... eccentric travelling companion

Given the ratio of velocities of a planet on an eliptical trajectory around a star in the pericenter (the point of the trajectory closest to the star) and in the apocenter (the point farthest from the star) $v_\mathrm{p}/v_\mathrm{a} = K = \sqrt{2}$, what's the numerical (relative) eccentricity ε of that trajectory?

Karel likes problems that deal with Kepler's 2nd law.

We'll use Kepler's 2nd law, which says that the area swept out per unit time is constant, so

$$w = \frac{v_\mathrm{a} a_\mathrm{a}}{2} = \frac{v_\mathrm{p} a_\mathrm{p}}{2}\,,$$

where a_a is the distance of the planet and sun in the apocenter and a_p is the distance in the pericenter. These distances can be expressed using its major semiaxis and eccentricity as $a_\mathrm{a} = a(1+\varepsilon)$ a $a_\mathrm{p} = a(1-\varepsilon)$. Substituting in the equation for the swept out areas and expressing the ratio of velocities, we obtain

$$v_\mathrm{a} a(1+\varepsilon) = v_\mathrm{p} a(1-\varepsilon) \quad\Rightarrow\quad \frac{v_\mathrm{p}}{v_\mathrm{a}} = K = \frac{1+\varepsilon}{1-\varepsilon}\,.$$

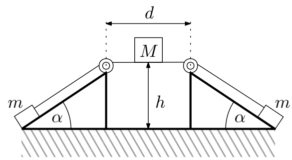
Now we have a relatively simple equation, containing only K and the unknown ε, from which we easily express

$$\varepsilon = \frac{K-1}{K+1} = \frac{\sqrt{2}-1}{\sqrt{2}+1} = 3 - 2\sqrt{2} \doteq 0.172\,.$$

The relative eccentricity of the trajectory is 0.172.

Karel Kolář
karel@fykos.cz

Problem FoL.25 ... cuboids up in the air



In the figure, you see a large cuboid of mass $M = 1\,\mathrm{kg}$, pulling two smaller cuboids of masses $m = M/4$ through pulleys along inclined planes with inclination angles $\alpha = 35^\circ$. Before the small cuboids reach the pulleys, the large cuboid will collide with the ground inelastically. What is the maximum height the small cuboids reach? We know the lengths marked in the figure: $d = 0.8\,\mathrm{m}$, $h = 0.6\,\mathrm{m}$; neglect the sizes of the cuboids and pulleys. Both the dynamic and static friction coefficients between the cuboids and the inclined plane are $f = 0.5$. We measure the height we are asking for with respect to the initial heights of the cuboids, the acceleration due to gravity is $g = 9.81\,\mathrm{m{\cdot}s^{-2}}$. There is no friction in the pulleys.

Mirek's ingenious construction.

The length of the rope between the mass M and a pulley is initially $d/2$. After the mass drops to the ground, its length will be

$$\sqrt{h^2 + \left(\frac{d}{2}\right)^2}\,.$$

At that time, a cuboid on an inclined plane will have moved up along the inclined plane by

$$l = \sqrt{h^2 + \left(\frac{d}{2}\right)^2} - \frac{d}{2}\,.$$

The potential energy of the large cuboid will decrease by Mgh, while the potential energy of a small cuboid increased by $mgl\sin\alpha$. The energetic balance of the system is

$$Mgh = \frac{1}{2}MV^2 + 2\left(\frac{1}{2}mv^2 + mgl\sin\alpha + W_\mathrm{t}\right)\,,$$

where v is the velocity of a small cuboid at the time of impact of the large cuboid, V is the velocity of the large cuboid just before the impact, and W_t is the energy dissipated due to friction during the movement of one small cuboid, given by

$$W_\mathrm{t} = mgfl\cos\alpha\,.$$

During the first phase of the movement, a small cuboid moves with the speed with which the length of the rope between the large cuboid and the respective pulley increases. From that, we obtain an expression for velocities of the large and small cuboids before the impact

$$v = V\frac{h}{\sqrt{h^2 + \left(\frac{d}{2}\right)^2}}\,.$$

Now, we're able to determine the velocity of a small cuboid

$$v = \sqrt{\frac{Mgh - 2mgl\sin\alpha - 2W_\mathrm{t}}{\frac{1}{2}\frac{Mh^2}{h^2 + \left(\frac{d}{2}\right)^2} + m}}\,.$$

With this velocity, the small cuboid is moving against gravity + friction with net acceleration $a = fg\cos\alpha + g\sin\alpha$. The distance travelled during movement with constant deceleration until stopping is $v^2/(2a)$, so together with the distance l travelled until the impact, we get the vertical distance by which a small cuboid moved

$$s = \left(l + \frac{Mh/2 - ml\sin\alpha - mfl\cos\alpha}{\left(m + \frac{Mh^2}{2\left(h^2 + (d/2)^2\right)}\right)(f\cos\alpha + \sin\alpha)}\right)\sin\alpha\,.$$

Numerically, $s \doteq 0.40\,\mathrm{m}$.

Miroslav Hanzelka
`mirek@fykos.cz`

Problem FoL.26 ... tuning

In the middle of a field, there's Pepa with a tuning fork emitting a sound with frequency $f_1 = 440\,\mathrm{Hz}$. You're driving away from Pepa with velocity $v = 70\,\mathrm{km{\cdot}h^{-1}}$. What will be the frequency f of the sound you hear? The speed of sound in air is $v_\mathrm{s} = 330\,\mathrm{m{\cdot}s^{-1}}$.

Olda made it.

The problem statement immediately suggests Doppler effect, which describes the change in received signal for a moving source or receiver. When the receiver, in this case us in a car, moves away from the source, it will detect a lower frequency than emitted from the source. Its magnitude can be found using the formula

$$f = f_1\frac{v_\mathrm{s} - v}{v_\mathrm{s}}\,,$$

where v_s is the speed of sound. After plugging in the numbers, we get $f = 414\,\mathrm{Hz}$.

Dominika Kalasová
`dominika@fykos.cz`

Problem FoL.27 ... aloha snackbar

The US President's airplane has range approximately $12\,600\,\mathrm{km}$. Once upon a time, it was hijacked by terrorists shortly after takeoff, with the president on board, and vanished from the radars. A thorough search of both land and ocean was launched immediately. What area (in km^2) has to be searched if the terrorists can't refuel anywhere? *Mikulas was kidnapped.*

First of all, we need to realise that at this range, curvature of Earth plays a significant role, so we can't work in Euclidean geometry. However, we can approximate Earth by a sphere. Even though formulae for the area of a spherical cap can be found easily, we'll use elemental integration. Let's denote the radius of Earth by R, the range of the plane by D and work in spherical coordinates with the center of Earth at the origin and the z axis passing through the place where the plane was kidnapped. Then, the area is given by a surface integral $\int \mathrm{d}S$. In our case, $\mathrm{d}S = R^2 \sin(\vartheta)\,\mathrm{d}\varphi\,\mathrm{d}\vartheta$, so we may write

$$S = \int \mathrm{d}S = R^2 \int_0^{2\pi} \int_0^{D/R} \sin\vartheta\,\mathrm{d}\vartheta\,\mathrm{d}\varphi = $$
$$= R^2 \int_0^{2\pi} 1 - \cos\frac{D}{R}\,\mathrm{d}\varphi = 2\pi R^2 \left(1 - \cos\frac{D}{R}\right) .$$

After plugging in our values, we obtain the correct area: approx. $3.6 \cdot 10^8\,\mathrm{km}^2$. If we used the formula πD^2, we'd get approx. $5 \cdot 10^8\,\mathrm{km}^2$, which is considerably different.

Mikuláš Matoušek
mikulas@fykos.cz

Problem FoL.28 ... burst

Into how many smaller droplets could a larger water droplet break, if it fell from a $h = 50\,\mathrm{cm}$ height? Consider all droplets to be spherical. The radius of the original droplet is $r = 1\,\mathrm{cm}$ (with its center of mass at height h) and the droplets it breaks into should have equal sizes. The surface tension of water is $\sigma = 73\,\mathrm{mN{\cdot}m^{-1}}$ and its density is $\varrho = 1.00\,\mathrm{g{\cdot}cm^{-3}}$. We're only looking for an estimate of an upper bound of the number of droplets based on the energetic balance at the beginning and at the end. *Karel was watching a falling droplet.*

According to the problem statement, the larger droplet has to break into n smaller ones. Assuming that the volume V of water is conserved, the radius of each of them will be

$$r_n = r\sqrt[3]{\frac{1}{n}} .$$

The potential energy that can be converted to surface energy is given by the formula

$$\Delta E = mgh = \frac{4}{3}\pi r^3 \varrho g h ,$$

where g is the acceleration due to gravity and $m = \varrho V$ the mass of one droplet. Here, we used the approximation $r \ll h$. The difference in surface energy between the situation when there's only one droplet and the situation with many smaller droplets is

$$\Delta E = \sigma \Delta S = \sigma\left(nS_n - S_1\right) = \sigma\left(4\pi r_n^2 n - 4\pi r^2\right) = 4\pi\sigma r^2\left(\sqrt[3]{n} - 1\right) .$$

Now, it's sufficient to equate those two expressions for energy

$$\frac{4}{3}\pi r^3 \varrho g h = 4\pi\sigma r^2\left(\sqrt[3]{n} - 1\right) \qquad \Rightarrow \qquad n = \left(\frac{\varrho g h r}{3\sigma} + 1\right)^3 \doteq 1.14 \cdot 10^7 .$$

The large droplet can break into at most 11 million smaller droplets. This is really just an upper bound, neglecting e.g. that the smaller droplets need to pass through non-spherical shapes while forming or that 100 % of the potential energy can't be converted to surface energy of the droplets.

Karel Kolář
karel@fykos.cz

Problem FoL.29 ... damped oscillation

There's a mass point that oscillates along the y axis. The oscillation is described by the equation $y = y_\mathrm{m}\mathrm{e}^{-bt}\sin(\omega t + \varphi_0)$, where y_m is the initial amplitude, $b = 0.05\,\mathrm{s}^{-1}$ is the damping coefficient, $\omega = 200\,\mathrm{Hz}$ is the angular frequency and φ_0 is the initial angular displacement. When will the maximal amplitude decrease below $y_\mathrm{m}/2$?

Karel was watching the oscillation of a strongly damped spring.

It's important to realise that the amplitude is given by the term $y_\mathrm{m}\mathrm{e}^{-bt}$ and multiplication by a trigonometric function gives oscillation, but doesn't affect the amplitude. Therefore, we're only solving the equation

$$y_\mathrm{m}\mathrm{e}^{-bt} = \frac{y_\mathrm{m}}{2}\,, \qquad \Rightarrow \qquad \mathrm{e}^{-bt} = \frac{1}{2}\,, \qquad \Rightarrow \qquad t = -\frac{\ln\frac{1}{2}}{b} = \frac{\ln 2}{b} \doteq 13.9\,\mathrm{s}\,.$$

The amplitude of the mass point is cut in half after 13.9 s.

Karel Kolář
karel@fykos.cz

Problem FoL.30 ... don't stick your fingers in!

A large ceiling fan may be approximated as a disk of radius $r = 10\,\mathrm{cm}$ and mass $M = 0.5\,\mathrm{kg}$, with three rods attached, pointing away from the centre of the disk. Each rod has length $l = 40\,\mathrm{cm}$ and mass $m = 0.2\,\mathrm{kg}$. The fan rotates with frequency $f = 1\,\mathrm{Hz}$. What work do we need to perform to stop the fan? All parts of the fan are homogeneous, neglect any friction.

Mirek was tickling the blades.

The approach to this problem is quite straightforward: we compute the rotational kinetic energy of the fan, which is equal to the work that needs to be performed to stop it. We need to know the moments of inertia of each part. The moment of inertia of a disk rotating around its axis is

$$I_0 = \frac{1}{2}Mr^2\,.$$

The moment of inertia of a rod rotating around the centre of the disk is

$$I_1 = \frac{1}{12}ml^2 + m\left(\frac{l}{2} + r\right)^2 = m\left(\frac{1}{3}l^2 + lr + r^2\right)\,.$$

The total moment of inertia of the fan (three rods plus one disk) is

$$I = I_0 + 3I_1 = \frac{1}{2}Mr^2 + m\left(l^2 + 3lr + 3r^2\right)\,.$$

The kinetic energy is computed as

$$E_\mathrm{k} = \frac{1}{2} I\omega^2 = \frac{1}{2} I(2\pi f)^2 = \left(Mr^2 + 2m(l^2 + 3lr + 3r^2)\right)\pi^2 f^2 \doteq 1.27\,\mathrm{J}\,.$$

In order to stop the fan, we need to perform 1.27 J of work.

Miroslav Hanzelka
mirek@fykos.cz

Problem FoL.31 ... dark planet

What's the radius of a sphere with Earth-like density $\varrho = 5.50\,\mathrm{g{\cdot}cm^{-3}}$, but escape velocity equal to the speed of light? We're interested in the first-order approximation, i.e. neglecting relativistic effects. Assume that the planet's density is constant (the planet is homogeneous). ***Give the result as a multiple of the radius of Earth*** $R_\mathrm{Z} = 6,371\,\mathrm{km}$.
Karel likes to talk about relativistic effects, but doesn't like actually solving relativistic problems.

The escape velocity of a planet with mass M and radius R can be looked up in formula lists or computed from equality of centrifugal and gravitational force. It's

$$v = \sqrt{\frac{2GM}{R}}\,,$$

where $G = 6.67\cdot 10^{-11}\,\mathrm{N{\cdot}kg^{-2}{\cdot}m^2}$ is the gravitational constant. The mass M can be expressed as $M = \varrho V$, where V is the volume of the planet – in our case, $V = 4\pi R^3/3$. We substitute that into the formula for escape velocity, which we set equal to the speed of light $v = c = 299,792,458\,\mathrm{m{\cdot}s^{-1}}$.

$$c = \sqrt{\frac{8\pi G R^3 \varrho}{3R}}\,, \qquad \Rightarrow \qquad R = \sqrt{\frac{3}{8\pi G\varrho}}\,c \doteq 1.7\cdot 10^{11}\,\mathrm{m} \doteq 27,000\,R_\mathrm{Z}\,.$$

Our mystical, hypothetical planet's radius would have to be 27,000 times larger than the radius of Earth.

Karel Kolář
karel@fykos.cz

Problem FoL.32 ... crash

The Moon is an approximately homogeneous sphere of mass $M = 7.3\cdot 10^{22}\,\mathrm{kg}$ and radius $R = 1,700\,\mathrm{km}$, rotating with a period $T = 27\,\mathrm{d}$. A megameteorite of mass $m = M/1\,000$ collides with the Moon's equator under an incidence angle $\vartheta = 45^\circ$ into the lunar soil, against the direction of rotation. The impact velocity is $v = 10\,\mathrm{km{\cdot}s^{-1}}$. How long will the lunar day be after the impact, if the debris from the meteorite spreads evenly over the surface of the Moon? The meteorite and the Moon are both made of the same material. Neglect mutual gravitational influence of the bodies before the impact. Give the result as a positive number in the units of (Earth's) days. *Mirek wanted the Moon to have a moon.*

The meteorite, which should collide with the Moon's surface under the angle $\vartheta = 45^\circ$, has an angular momentum with respect to the Moon's surface

$$L_\mathrm{m} = \eta M v R \frac{\sqrt{2}}{2}\,,$$

where we denoted $\eta = 1/1\,000$. Since it is an impact against the original direction of rotation, this angular momentum has to be subtracted from the original angular momentum of the Moon, so

$$L' = L - L_\mathrm{m}\,,$$

where L' is the new angular momentum of the Moon and L is its original ang. momentum. The moment of inertia of the spherical Moon around its center is

$$I = \frac{2}{5}MR^2\,.$$

After the impact,

$$I' = \frac{2}{5}\left(M + \eta M\right)\left(R\sqrt[3]{1+\eta}\right)^2\,.$$

Since angular momentum can be written as $L = I\omega$, where ω is the angular velocity $\omega = 2\pi/T$, the new angular velocity ω' will be

$$\omega' = \frac{I}{I'}\omega - \frac{L_\mathrm{m}}{I'}\,,$$

which can be written using rotation periods as

$$T' = \frac{1}{\frac{I}{I'}\frac{1}{T} - \frac{L_\mathrm{m}}{2\pi I'}}\,.$$

After simplifying and substituting, we obtain the new period

$$T' = \frac{(1+\eta)^{5/3}}{\frac{1}{T} - \frac{5\eta}{4\pi}\frac{v}{R}\frac{\sqrt{2}}{2}} \doteq -9.5\,\mathrm{d}\,.$$

The new period is 9.5 d, but the Moon is rotating in the opposite direction.

Miroslav Hanzelka
mirek@fykos.cz

Problem FoL.33 ... just a little closer

There are two hooks on two opposite walls with distance $d = 1\,\mathrm{m}$. On one hook, there's a charge $q = 1\,\mu\mathrm{C}$. On the other hook, there's no charge, but a thin non-conducting massless rope of length $l = d$, with a ball with mass $m = 2\,\mathrm{g}$ and charge $-q$ on its other end. Find the smallest angle (in °) between the rope and the wall such that the ball is at rest.

Mirek was reaching for his notebook.

Let's denote the unknown angle by α, the angle between the line connecting the ball with the other hook and the vertical by β. The ball splits the distance between the walls into two parts of lengths $l\sin\alpha$ and $l(1-\sin\alpha)$. The vertical distance of the ball from the hooks is $l\cos\alpha$. The angle β can then be expressed using the angle α as

$$\mathrm{tg}\,\beta = \frac{1-\sin\alpha}{\cos\alpha}\,.$$

The electromagnetic force can be written in the form $F_\mathrm{e} = a/r^2$, where

$$a = \frac{q^2}{4\pi\varepsilon_0}$$

and r is the distance of the ball from the other hook. For the ball to be at rest, the net force of $F_g = mg$ and F_e has to point in the direction of the rope. This geometric fact can be expressed by the equation

$$\operatorname{tg}\alpha = \frac{F_\mathrm{e}\sin\beta}{F_g - F_\mathrm{e}\cos\beta} = \frac{\operatorname{tg}\beta}{\dfrac{mgr^2}{a\cos\beta} - 1}$$

We may compute from the Pythagorean theorem

$$r = l\sqrt{2(1-\sin\alpha)}$$

and also

$$\cos\beta = \frac{l\cos\alpha}{r} = \frac{\cos\alpha}{\sqrt{2(1-\sin\alpha)}}\,.$$

Substituting for $\cos\beta$ and $\operatorname{tg}\beta$ in the expression for $\operatorname{tg}\alpha$, we get after several simplifications and substituting for r

$$1 - \operatorname{tg}\alpha\frac{mgl^2}{a}\bigl(2(1-\sin\alpha)\bigr)^{3/2} = 0\,.$$

The smallest positive root of this expression is $\alpha \doteq 0.239 \doteq 13.7^\circ$.

Miroslav Hanzelka
mirek@fykos.cz

Problem FoL.34 ... chain reaction II

We placed a large number of identical ice cuboids with square bases of side $a = 2\,\mathrm{cm}$ in a line on an ice rink. The distance between centers of any two successive cuboids is $d = 5\,\mathrm{cm}$. Each cuboid has mass $m = 20\,\mathrm{g}$ and the friction coefficients (both static and dynamic) are $f = 0.03$. The first cuboid in the line is given a velocity $v_0 = 1\,\mathrm{m{\cdot}s^{-1}}$ in order to start a series of central collisions of the cuboids. Which cuboid will be the last one to move? All collisions are perfectly inelastic, the cuboids don't tilt and are placed in such a way that they have parallel sides. The acceleration due to gravity is $g = 9.81\,\mathrm{m{\cdot}s^{-2}}$.

Mirek was strolling through the St. Wenceslas square again.

During the first cuboid's movement, there's a frictional force acting on it, so its acceleration is $a_0 = fg$ opposite to the direction of its movement. Denoting the distance of successive cuboids' near sides by $\delta = d - a$; over this distance, the cuboid's velocity decreases from v_0 to $\sqrt{v_0^2 - 2a_0\delta}$. After a perfectly inelastic collision, the mass is doubled (the cuboids stick together) and due to momentum conservation, the velocity after the first collision will be

$$v_1 = \frac{1}{2}\sqrt{v_0^2 - 2a_0\delta}\,.$$

In the next collisions, the mass will keep increasing, so the velocity is multiplied by $n/(n+1)$ after the n-th collision. Together with the formula for the decrease of velocity between the collisions, we find the velocity after the n-th collision to be

$$v_n = \frac{1}{n+1}\sqrt{v_0^2 - 2a_0\delta\sum_{k=1}^{n} k^2} = \frac{1}{n+1}\sqrt{v_0^2 - \frac{n(n+1)(2n+1)fg\delta}{3}}\,.$$

The largest integer n, for which the above formula still has physical meaning, is $n = 5$. The last cuboid to move will therefore be the sixth one.

Miroslav Hanzelka
`mirek@fykos.cz`

Problem FoL.35 ... Newton's rings

Olda was watching Newton's glass. Compute the distance of the 2nd minimum of the interference circles from the center, if we know that the light incident on it has wavelength $\lambda = 500\,\mathrm{nm}$, the refraction index of glass is $n_\mathrm{s} = 1.5$ and the refraction index of air is $n_\mathrm{v} = 1$. The radius of curvature of the upper glass is $r = 20\,\mathrm{cm}$. *Olda was looking at Newton's ring.*

The phase difference of interfering rays is caused by travelling through air from the upper glass to the bottom one and back, and from a phase shift of π when reflected from an optically denser medium – in total, we get

$$\delta\varphi = 2yn_\mathrm{v}k_0 - \pi\,,$$

where k_0 is the wave number (we have $k_0 = 2\pi/\lambda$) and y is the height of the upper, round glass above the lower one.

For the following calculations, we need to determine y depending on the distance from the middle of the round glass. From the Pythagorean theorem, we have

$$x^2 + (y-r)^2 = r^2\,,$$
$$x^2 + y^2 - 2ry = 0\,.$$

In the approximation $y \ll r$, we may neglect the term y^2. Then, the formula for y is

$$y = \frac{x^2}{2r}\,.$$

After substituting in the expression for $\delta\varphi$, we have

$$\delta\varphi = k_0 n_\mathrm{v}\frac{x^2}{r} - \pi\,.$$

The interference minima happen when the phase difference of interfering rays is

$$\delta\varphi = 2m\pi - \pi\,,$$

where m is the number of the minimum (in the center, we have the 0^th minimum). Combining these equations, we get the condition for calculating the minimum

$$x = \sqrt{\frac{mr\lambda}{n_\mathrm{v}}} = \sqrt{2r\lambda} \doteq 4.47\cdot 10^{-4}\,\mathrm{m}\,.$$

In the solution, we assumed that the rays only refract negligibly. We can see that $x \ll r$, so the assumption was reasonable and we also get $y \ll r$.

Oldřich Holcner
holcner@fykos.cz

Problem FoL.36 ... short circuit

There's a simple electric circuit with two elements: a DC source with voltage $U = 1\,\mathrm{V}$ and internal resistance $R = 1\,\Omega$, and a coil with $N = 200$ turns of copper winding (circular cross-section), length $l = 30\,\mathrm{cm}$, inner radius $r_1 = 1.90\,\mathrm{cm}$ and outer radius $r_2 = 2.05\,\mathrm{cm}$. What's the maximum current passing through the coil after we switch on the source? The temperature in the room is $20\,^{\circ}\mathrm{C}$, neglect any heating of the winding.

Punchline redacted to avoid people whining about sexism.

Even though the source has internal resistance, we need to take into account the resistance of the coil's winding as well:

$$R_\mathrm{L} = \varrho_\mathrm{Cu} \frac{4 l_\mathrm{d}}{\pi (r_2 - r_1)^2}\,,$$

where l_d is the length of the wire, $r_2 - r_1$ is the diameter of the wire and $\varrho_\mathrm{Cu} = 1.68 \cdot 10^{-8}\,\Omega{\cdot}\mathrm{m}$ is copper resistivity at the given temperature. The length of the wire is anything but equal to l; in addition, the wire isn't wound in a circle, but slanted a bit. We can view it this way: the length of the wire along the axis of the coil is l and the length "normal to the axis" is $\pi(r_1 + r_2)N$, so the total length of the wire

$$l_\mathrm{d} = \sqrt{l^2 + \left(\pi(r_1 + r_2)N\right)^2}$$

according to the Pythagoren theorem. (In this case, only the latter term is relevant and $l_\mathrm{d} \approx \pi(r_1 + r_2)N$.)

Next, we know that if there's direct current passing through the coil, there's no voltage induced in it and the coil acts as a simple wire. Intuitively, it would make sense for the current to be maximum with direct current, which gives

$$I_\mathrm{max} = \frac{U}{R + R_\mathrm{L}} \approx \frac{U}{R + \varrho_\mathrm{Cu} \frac{4(r_1 + r_2)N}{(r_2 - r_1)^2}} \doteq 0.809\,\mathrm{A}\,.$$

The exact expression for $I(t)$ after an instantaneous switch-on can be calculated from the differential equation $U = (R + R_\mathrm{L})I + L\dot{I}$, whose solution for $I(0) = 0$ has the form $I = A(1 - \mathrm{e}^{-\lambda t})$; plugging it back in the diff. equation gives $\lambda = (R + R_\mathrm{L})/L$ and $A = U/(R + R_\mathrm{L})$. The current will be approaching the previously computed maximum I_max, but never reach (or exceed) it.

Jakub Šafin
xellos@fykos.cz

Problem FoL.37 ... exoplanet

An exoplanet of radius $r_\mathrm{p} = 8.0 \cdot 10^3\,\mathrm{km}$ is in circular orbit around a distant star with mass $M = 1.0 \cdot 10^{30}\,\mathrm{kg}$. The orbit and the observer are coplanar. A transit of the planet across the star *was observed. The time from the first decrease in brightness until brightness returned to its initial value was* $T = 2.0\,\mathrm{hours}$ *and the maximum decrease in brightness was* $0.3\,\%$. *Determine the orbital radius.* *Filip read about newly discovered exoplanets.*

Let's denote the radius of the star by R. If we look at the star and the planet from above, we can see that since the star is very distant, the light rays we see from Earth are almost parallel. The brightness starts decreasing when the planet's edge first touches the line between the Earth and an edge of the star and returns to its original value again when it touches the line between the Earth and the other edge of the star from the outside. We can assume that the radius of the star is much smaller than the radius of the planets orbit and much larger than the radius of the planet. Then, the distance the planet has to travel between these events is $Tv \approx 2R + 2r \approx 2R$. The planet moves at a constant orbital velocity

$$v = \sqrt{\frac{GM}{r}}\,,$$

The planet blocks a part of light proportional to the blocked surface area of the star as visible from Earth, so $p = r_p^2/R^2$. Putting it all together:

$$T\sqrt{\frac{GM}{r}} = 2\frac{r_p}{\sqrt{p}}$$

$$r = \frac{pT^2GM}{4r_p^2}\,.$$

Therefore, $r \doteq 4.05 \cdot 10^{10}\,\mathrm{m}$. By evaluating the orbital velocity at this distance we can easily verify that our assumption was correct; it takes the planet $0.39\,\mathrm{s}$ to travel the distance $2r$, which is only $0.005\,\%$ of the total transit time.

Filip Ayazi
filip@fykos.cz

Problem FoL.38 ... tse-tse

Little rabbit has a prescription for sleeping pills. When he takes 1/3 of a pill, he sleeps for 5 hours. When he takes 2/3 of a pill, he sleeps for 9 hours. Once, little rabbit went nuts and took 100 pills. How many hours did he sleep, if we know that little rabbit sleeps only when the medicine's concentration in blood exceeds a certain value? Assume that the rate at which the medicine is broken down is proportional to its concentration. Neglect any medicine left over in blood from the previous days or little rabbit's death. *Mikulas couldn't sleep.*

Solving the differential equation

$$\frac{\mathrm{d}c}{\mathrm{d}t} = -\alpha c\,,$$

we find that the medicine's concentration has to decrease exponentially according to the formula $c = c_0 \exp(-\alpha t)$ with an unknown constant α. We'll determine that based on our knowledge of little rabbit's sleep times. Let's denote the concentration after taking one pill by c_1, the concentration, at which little rabbit wakes up, by c_x, and consider all times to be in hours (so α will be in h^{-1}). We know that

$$\frac{1}{3}c_1\mathrm{e}^{-5\alpha} = c_x\,,$$
$$\frac{2}{3}c_1\mathrm{e}^{-9\alpha} = c_x\,.$$

Comparing the two equations and expressing the exponent, we get

$$\alpha = \frac{\ln 2}{4}\,\mathrm{h}^{-1}\,.$$

Comparing with the exponential containing the unknown time t, we get the equation

$$100\,\mathrm{e}^{-\frac{\ln 2}{4}t} = \frac{1}{3}\mathrm{e}^{-5\frac{\ln 2}{4}}\,,$$

whose solution is

$$t = \frac{5\frac{\ln 2}{4} + \ln 300}{\frac{\ln 2}{4}}\,,$$

numerically approx. 38 hours. That's not so much if we consider that little rabbit took many times more than a day's dose. In practice, of course, liver enzymes get saturated at high concentrations and the breakdown rate stops being dependent on concentration, decreasing linearly instead.

Mikuláš Matoušek
mikulas@fykos.cz

Problem FoL.39 ... throwing peas at the wall

Mirek was sitting on an office chair with wheels at work, looking at a wall deep in thoughts. In order to do something more useful, he took a sack of peas and started throwing them at a wall. There are $n = 3,600$ peas in the sack, each of them weighs $m_\mathrm{h} = 0.2\,\mathrm{g}$ and Mirek is throwing them with a frequency $f = 0.5\,\mathrm{s}^{-1}$. The peas are thrown horizontally with velocity $v_\mathrm{h} = 10\,\mathrm{m{\cdot}s}^{-1}$. What will Mirek's velocity be after emptying the sack? The mass of Mirek + chair is $m_\mathrm{M} = 60\,\mathrm{kg}$, the chair moves without friction. *Mirek needed to move on.*

The principle is the same as for acceleration of a rocket, so let us use the same approach as when deriving Tsiolkovsky's equation (we will consider the throwing of peas to be continuous). Let Mirek, the chair and the sack in total weigh $m(t)$ at time t and move in the x-direction, with initial conditions $x(0) = 0$, $\dot{x}(0) = 0$, $m(0) = m_\mathrm{M} + nm_\mathrm{h}$. Mirek's momentum along with the chair and sack change according to the formula

$$\frac{\mathrm{d}(m\dot{x})}{\mathrm{d}t} = \frac{\mathrm{d}m}{\mathrm{d}t}\dot{x} + m\ddot{x}\,.$$

In order to work with the net velocity of the system, we have to add the momentum of the thrown peas

$$-\frac{\mathrm{d}m}{\mathrm{d}t}v_\mathrm{o}\,,$$

where the velocity v_o of a pea is measured in the rest frame, so $v_\mathrm{o} = \dot{x} - v_\mathrm{h}$ (we consider all velocities to be positive). The net momentum change has to be zero, so

$$0 = \frac{\mathrm{d}m}{\mathrm{d}t}\dot{x} + m\ddot{x} - \frac{\mathrm{d}m}{\mathrm{d}t}v_\mathrm{o}\,,$$

from which we get

$$m\ddot{x} = -\frac{\mathrm{d}m}{\mathrm{d}t}\left(\dot{x} - v_\mathrm{o}\right) = -v_\mathrm{h}\frac{\mathrm{d}m}{\mathrm{d}t}\,.$$

Integrating the right hand side with respect to mass gives Mirek's velocity

$$v_\mathrm{M} = v_\mathrm{h}\ln\frac{m(0)}{m(n/f)}\,,$$

where $t = n/f$ is the time it takes Mirek to empty the sack. After plugging in the values,

$$v_\mathrm{M} = v_\mathrm{h}\ln\frac{m_\mathrm{M} + nm_\mathrm{h}}{m_\mathrm{M}} \doteq 0.1193\,\mathrm{m{\cdot}s^{-1}}\,.$$

If you think it's a small number, try to compute the distance Mirek passed.

Let us also note that if we realise at the start that $m_\mathrm{M} \gg nm_\mathrm{h}$, we may linearise the problem and obtain the result in the form

$$v_\mathrm{M} = v_\mathrm{h}\frac{nm_\mathrm{h}}{m_\mathrm{M} + nm_\mathrm{h}/2}\,,$$

which is close to the Taylor series of the formula above.

Miroslav Hanzelka
mirek@fykos.cz

Problem FoL.40 ... wire

There's a thin, perfectly black wire with circular cross-section of radius $r = 0.3\,\mathrm{mm}$ in thermal equilibrium at room temperature ($T_0 = 20\,\mathrm{^\circ C}$) in vacuum. Under these conditions, the resistance of the wire per unit length is $R_0 = 5\,\Omega/\mathrm{m}$. If the thermal coefficient of resistance is $\alpha = 0.004\,\mathrm{K^{-1}}$, determine the current that must pass through the wire so that its temperature stabilised at $220\,\mathrm{^\circ C}$. *Filip burnt his finger.*

As the wire is in vacuum, the only way it can exchange energy with its environment is through radiative transfer. The radiated power is given by the Stefan-Boltzmann law $P = 2\pi rl\sigma\left(T^4 - T_0^4\right)$, where l is the length of the wire and σ is the Stefan-Boltzmann constant. Then, we only have to equate this power to the power delivered by electric current $P = I^2R = I^2R_0l\left(1 + \alpha(T - T_0)\right)$, from which

$$I = \sqrt{\frac{2\pi r\sigma\left(T^4 - T_0^4\right)}{R_0\left(1 + \alpha\left(T - T_0\right)\right)}}$$

and plugging in the given values, $I \doteq 0.78\,\mathrm{A}$.

Filip Ayazi
filip@fykos.cz

Problem FoL.41 ... hydrophobe

There's a water droplet on the horizontal part of an umbrella. The contact angle between the droplet and the umbrella is quite large, $\vartheta_c = 120^\circ$. The surface tension between water and air is $\sigma_{lg} = 73\,\mathrm{mN{\cdot}m^{-1}}$. We add a bit of detergent to the droplet, decreasing the surface tension to $\sigma'_{lg} = 56\,\mathrm{mN{\cdot}m^{-1}}$ and the contact angle to $\vartheta'_c = 90^\circ$. How much did the surface tension between the surface of the umbrella and the liquid have to change? Compute the result in units $\mathrm{mN{\cdot}m^{-1}}$! *Mirek was inspecting his broken umbrella.*

The forces of surface tension act between the liquid and the surface of the umbrella (surface tension σ_{sl}), between air and the umbrella (σ_{sg}) and between liquid and air (σ_{lg}). For the forces to be balanced, their vector sum has to be zero. Since σ_{sg} and σ_{sl} act in opposite directions along the surface of the umbrella and the force between air and liquid is inclined by ϑ_c from that surface, the following scalar equalities for liquid without/with the detergent hold

$$\sigma_{sg} - \sigma_{sl} - \sigma_{lg}\cos\vartheta_c = 0\,,$$
$$\sigma'_{sg} - \sigma'_{sl} - \sigma'_{lg}\cos\vartheta'_c = 0\,.$$

The surface tension between the umbrella and the air doesn't change, so $\sigma'_{sg} = \sigma_{sg}$. Subtracting the equations, we easily find the change in surface tension

$$\sigma_{sl} - \sigma'_{sl} = \sigma'_{lg}\cos\vartheta'_c - \sigma_{lg}\cos\vartheta_c \doteq 36.5\,\mathrm{mN{\cdot}m^{-1}}\,.$$

The surface tension between the umbrella material and the liquid decreased after adding the detergent by $36.5\,\mathrm{mN{\cdot}m^{-1}}$.

Miroslav Hanzelka
mirek@fykos.cz

Problem FoL.42 ... distributed

When describing plasma in magnetic field, it's often useful to work in cylindrical coordinates related to magnetic field lines. In these coordinates, we can model an anisotropic velocity distribution using a so-called bi-Maxwell distribution function

$$f(v_\perp, v_\parallel) = \exp\left(-\frac{\left(v_\parallel - \mu_\parallel\right)^2}{2\sigma_\parallel^2}\right)\exp\left(-\frac{(v_\perp - \mu_\perp)^2}{2\sigma_\perp^2}\right)\frac{1}{(2\pi)^{3/2}\sigma_\perp\sigma_\parallel}\,.$$

The indices $\parallel$ and $\perp$ denote components parallel to the magnetic field and perpendicular to it; the angular component has been removed by integration (due to axial symmetry). The parameters of the velocity distribution are known to be $\sigma_\parallel = 10^6\,\mathrm{m{\cdot}s^{-1}}$, $\sigma_\perp = 4\cdot 10^5\,\mathrm{m{\cdot}s^{-1}}$ and $\mu_\parallel = 4\cdot 10^4\,\mathrm{m{\cdot}s^{-1}}$, $\mu_\perp = 3\cdot 10^4\,\mathrm{m{\cdot}s^{-1}}$. Determine the mean velocity of a particle described by the distribution $f(v_\perp, v_\parallel)$. *Mirek submitted a practical problem.*

If we draw or imagine the said distribution, it's immediately clear that it's composed of two independent distributions in the axis parallel to the field and the axis perpendicular to it. In the graph with axes $v_\perp$ and $v_\parallel$, the maximum of the function $f(v_\perp, v_\parallel)$ (and, due to symmetry, the mean value as well) lies at $[\mu_\parallel, \mu_\perp]$. We obtain

$$\mu = \sqrt{\mu_\parallel^2 + \mu_\perp^2} = 5\cdot 10^4\,\mathrm{m{\cdot}s^{-1}}$$

simply by using the Pythagorean theorem. The mean velocity of particles is $\mu = 5 \cdot 10^4\,\mathrm{m\cdot s^{-1}}$.

Miroslav Hanzelka
mirek@fykos.cz

Problem FoL.43 ... merry-go-round

There's a box of mass $m = 5\,\mathrm{kg}$ and negligible size standing on a merry-go-round. The box is $r = 10\,\mathrm{m}$ far from the center of the merry-go-round; the static friction coefficient between them is $f_0 = 1$. The merry-go-round starts rotating from rest with linearly increasing angular acceleration $\varepsilon = kt$, where $k = 1\,\mathrm{s^{-3}}$. How long since the start of the rotation will it take for the box to fly away from its initial position? *Kuba wanted a problem where you need Coriolis...*

The friction force F acts in the reference frame of the merry-go-round, where its magnitude is just big enough to keep the box at rest (standing on the merry-go-round). The net force acting against friction is the vector sum of the centrifugal force $F_\mathrm{c} = mr\omega^2$ in the radial direction and the Euler force $F_\mathrm{E} = mr\varepsilon$ in the angular direction. We can thus write a force balance in the form

$$F = \sqrt{F_\mathrm{c}^2 + F_\mathrm{E}^2} = mr\sqrt{\omega^4 + \varepsilon^2} = mkr\sqrt{t^2 + \frac{k^2t^8}{16}}\,.$$

The maximum value of the friction force is

$$F \leq mgf_0\,.$$

For the critical time τ, we obtain a quartic equation

$$\frac{k^2\tau^8}{16} + \tau^2 - \left(\frac{gf_0}{kr}\right)^2 = 0\,,$$

which can be solved numerically. The equation has only one positive root, which is $\tau \doteq 0.96\,\mathrm{s}$.

Jakub Dolejší
krasnykuba@fykos.cz

Problem FoL.44 ... twin paradox

One of two twins leaves in a rocket for a planet $s = 5\,\mathrm{ly}$ far from the Earth, while the other twin stays at their home planet. The rocket moves with velocity $v = 0.2c$. At half the distance s, the cosmonaut twin enters a fast module, which starts moving with respect to the original rocket with velocity $u = 0.1c$ towards the original goal of the rocket. How many years younger will the cosmonaut twin be compared to the twin at the home planet? Neglect any effects caused by acceleration or deceleration of the space vessels (as if they only passed by each other and gravity did not exist). In order to avoid trouble with relativity of simultaneity, compare the times spent by each twin waiting for the arrival of the fast module at the goal.
All special relativity problems are washed out.

Let's measure the distances in light years, the times in years and velocities in multiples of light speed.

The first half of the trip in the home planet's reference frame takes

$$t_1 = \frac{s}{2v}\,.$$

Since the first half of the trip is local in the rocket's reference frame, we may use the simple formula for time dilation in the rocket's reference frame

$$t'_1 = \frac{s}{2v}\sqrt{1 - v^2}\,.$$

In order to be able to use the same formula[1] for the second half of the trip, we need to transform the velocity u from the rocket's reference frame to the reference frame of the home planet. We get

$$w = \frac{u + v}{1 + uv}\,.$$

Since the second half of the trip is local with respect to the module, we may write

$$t_2 = \frac{s}{2w}\,,$$

$$t'_2 = \frac{s}{2w}\sqrt{1 - w^2} = \frac{s}{2}\frac{\sqrt{1 - v^2}\sqrt{1 - u^2}}{v + u}\,.$$

Now, we can express the twins' years in the form

$$\tau = \frac{s}{2}\left(\frac{1}{v} + \frac{1 + uv}{u + v}\right)\,,$$

$$\tau' = \frac{s}{2}\left(\frac{\sqrt{1 - v^2}}{v} + \frac{\sqrt{1 - v^2}\sqrt{1 - u^2}}{u + v}\right)\,.$$

The age difference between the twins is therefore

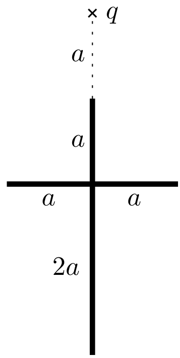
$$\Delta\tau = \frac{s}{2}\left(\frac{1 - \sqrt{1 - v^2}}{v} + \frac{1 + uv - \sqrt{1 - v^2}\sqrt{1 - u^2}}{u + v}\right) \doteq 0.63\,\mathrm{yr}\,.$$

The sought for difference in the twins' ages is 0.63 years.

Jakub Dolejší
krasnykuba@fykos.cz

[1] The duration of the second half of the trip could be determined alternatively by viewing it from the rocket, accounting for length contraction (from the rocket to the goal planet – speed v) and then for time dilation (from the rocket to the module – length u).

Problem FoL.45 ... vampirectrical



There's a charged non-conducting cross and a point charge, placed as shown in the figure. We know $a = 20\,\mathrm{cm}$, $q = 1\,\mathrm{\mu C}$ and the linear charge density of the cross $\lambda = q/a$. What's the magnitude of the electrostatic force acting on the charge? *Mirek was defending himself from diabolical problems.*

Let's denote the direction of the shorter rod by x, the direction of the longer rod by z and the normal to the figure by y. It's clear that there's no force acting in the y-direction. The origin of the coordinate system is the point of contact of the rods.

Let's take a look at the shorter rod. Since the testing charge is placed in the middle above it, the force acting in the x-direction cancels out and we only have the z-component of the force remaining. An element of length $\mathrm{d}x$ in the point $[x, 0, 0]$ contributes to the intensity of the electric field by

$$\mathrm{d}E_{1z} = \frac{1}{4\pi\varepsilon_0} \frac{\lambda\,\mathrm{d}x \cos\alpha}{(2a)^2 + x^2}\,,$$

where $\cos\alpha$ projects the intensity vector to the z-axis and

$$\cos\alpha = \frac{2a}{\sqrt{(2a)^2 + x^2}}\,.$$

Overall, E_{1z} from the shorter rod can be found by integrating

$$E_{1z} = \int_{-a}^{a} \mathrm{d}E_{1z} = \int_{-a}^{a} \frac{1}{4\pi\varepsilon_0} \frac{2\lambda a\,\mathrm{d}x}{(4a^2 + x^2)^{3/2m}} = \frac{\lambda}{4\pi\varepsilon_0} \left[\frac{x}{2a\sqrt{4a^2 + x^2}}\right]_{-a}^{a} = \frac{q}{4\pi\varepsilon_0 a^2} \frac{1}{\sqrt{5}}\,.$$

In the case of the vertical rod, the situation is even simpler. Due to symmetry, the only non-zero component is E_{2z} again and the element with distance z contributes to the intensity of the el. field by

$$\mathrm{d}E_{2z} = \frac{1}{4\pi\varepsilon_0} \frac{\lambda\,\mathrm{d}z}{z^2}\,.$$

Integrating gives

$$E_{2z} = \int_{a}^{4a} \mathrm{d}E_{2z} = \int_{a}^{4a} \frac{1}{4\pi\varepsilon_0} \frac{\lambda\,\mathrm{d}z}{z^2} = \frac{1}{4\pi\varepsilon_0} \left[-\frac{\lambda}{z}\right]_{a}^{4a} = \frac{q}{4\pi\varepsilon_0 a^2} \frac{3}{4}\,.$$

The electrostatic force is

$$F = q(E_{1z} + E_{2z}) = \frac{q^2}{4\pi\varepsilon_0 a^2} \left(\frac{3}{4} + \frac{1}{\sqrt{5}}\right)\,.$$

After plugging in the given values, we get $F \doteq 0.27\,\mathrm{N}$.

Miroslav Hanzelka
mirek@fykos.cz

Problem FoL.46 ... focus

On one side of a thin biconvex lens with identical radii of curvature $R = 20\,\mathrm{cm}$ is a spherical mirror placed coaxially in such a way that a ray that passes through the lens is reflected back towards the lens. What is the (positive) focal length of such a system? The refractive index of glass is $n = 1.5$.

Damn optics again!

A focus is a place where collinear rays gather, so we are solving a pair of thin lens equations for light from a source at $a_0 = \infty$.

A thin lens has optical power

$$\varphi = \frac{1}{f} = \frac{2}{R}(n-1)$$

and a spherical mirror has focal length

$$f' = \frac{R}{2}\,.$$

One by one, we go through the chain of images: image of the source a_0 using the lens is at a_1, then its image using the mirror is at a_2 and then the image from a_2 again using the lens is at a_3,

$$\varphi = \frac{1}{a_1}\,,$$
$$\frac{2}{R} = -\frac{1}{a_1} + \frac{1}{a_2}\,,$$
$$\varphi = -\frac{1}{a_2} + \frac{1}{a_3}\,,$$

where we used the following sign convention: an object before the lens and an image behind the lens have positive positions, and both an object and an image before a mirror (real) have positive positions.

Solving this system of equations, we obtain the focal length of the whole system in the form of the position of the final image a_3

$$f = a_3 = \frac{R}{4n-2} = 5\,\mathrm{cm}\,.$$

Focal length of the whole system is $5\,\mathrm{cm}$.

Jakub Dolejší
krasnykuba@fykos.cz

Problem FoL.47 ... Sisyphean labour

Sisyphos is standing at the bottom of a hemispherical hole of radius $R = 100\,\mathrm{m}$. His task is to push an approximately spherical, bumpy rock of mass $M = 1\,000\,\mathrm{kg}$ and radius $r = 50\,\mathrm{cm}$. The effective rolling friction coefficient of the rock is $\xi = 4\,\mathrm{cm}$. Sisyphos is pushing the rock up with negligible constant velocity while exerting a force F_s, horizontally towards the center of mass of the rock. Sisyphos runs out of stamina after performing work $W_\mathrm{s} = 200\,\mathrm{kJ}$. By how much will he lift the center of mass of the rock before it happens? Neglect changes in potential energy of Sisyphos, the acceleration due to gravity is $g = 9.81\,\mathrm{m{\cdot}s^{-2}}$.

Mirek tried to resolve all unread e-mails.

Let us view the situation in the plane in which Sisyphos pushes the rock. The rolling friction force, acting at the point of contact of the rock and the earth against the direction of the rock's movement, can be expressed as

$$F_\mathrm{o} = F_\mathrm{n}\frac{\xi}{r}\,,$$

where F_n is the normal force, pushing the rock down towards the ground. It is composed of gravity and Sisyphos's force, so

$$F_\mathrm{n} = F_g\cos\alpha + F_\mathrm{s}\sin\alpha\,,$$

where α is the angle between the direction of the rock's movement and the vertical plane, and $F_g = Mg$. Using the angle α, we can also express the change in potential energy of the rock

$$\Delta E_\mathrm{p} = Mg(R-r)(1-\cos\alpha) \approx MgR(1-\cos\alpha)\,.$$

The total work performed by Sisyphos is

$$W_\mathrm{s} = W_\mathrm{o} + Mg(R-r)(1-\cos\alpha)\,,$$

where W_o is the work performed by friction. In order to determine the magnitude of the friction force, we will use the scalar force balance

$$F_\mathrm{s}\cos\alpha = F_g\sin\alpha + F_\mathrm{o}\,,$$

which gives

$$F_\mathrm{o} = \frac{\frac{\xi}{r}Mg}{\cos\alpha - \frac{\xi}{r}\sin\alpha}\,.$$

after plugging into the equation for rolling resistance. Now, we will assume $\cos\alpha \gg \xi/r\sin\alpha$ (which we will verify later), so we may write

$$F_\mathrm{o} \approx \frac{\xi}{r}Mg\frac{1}{\cos\alpha}\,.$$

The work of the rolling friction then is

$$W_\mathrm{o} \approx \frac{\xi}{r}RMg\int_0^\alpha \frac{\mathrm{d}\varphi}{\cos\varphi} = \frac{\xi}{r}RMg\ln\left(\mathrm{tg}\,\alpha + 1/\cos\alpha\right)\,.$$

The formula for the angle α, at which Sisyphos runs out of stamina, can then be written in an approximate form ($r \ll R$)

$$\frac{W_\mathrm{s}}{MgR} \approx \frac{\xi}{r}\ln\left(\mathrm{tg}\,\alpha + 1/\cos\alpha\right) + 1 - \cos\alpha\,.$$

Now, all that remains is to numerically determine the physically meaningful root $\alpha \doteq 0.566$ and (without any approximations now) compute

$$\Delta y = (R-r)(1-\cos\alpha) \doteq 15.5\,\mathrm{m}\,.$$

Sisyfos will roll the rock up to the height $\Delta y \doteq 15.5\,\mathrm{m}$. We may verify that the approximation $\cos\alpha \gg \xi/r\sin\alpha$ was good, because $0.84 \gg 0.04$. A more accurate computation (integrating without approximations) gives the result $\Delta y \doteq 14.4\,\mathrm{m}$, which is also accepted, of course, but it cannot be written in a nice form. The result $\Delta y \doteq 20\,\mathrm{m}$, obtained by neglecting friction, was not accepted.

Miroslav Hanzelka
mirek@fykos.cz

Problem FoL.48 ... spin it

There is a bicycle on a bicycle stand. We know that its rear wheel's moment of inertia around its axis is $I = 0.5\,\mathrm{kg{\cdot}m^2}$ and its radius is $r = 30\,\mathrm{cm}$. We speed up the wheel by repeatedly hitting it with a hand tangentially on its circumference in the direction of its rotation. The speed of each hit in the rest reference frame is $w = 5\,\mathrm{m{\cdot}s^{-1}}$, the mass of the hand is $m = 1.5\,\mathrm{kg}$, the hand loses all its velocity in the reference frame of the point of impact after the impact *(the hand is moving around an elbow, but we neglect rotation of the forearm). What will the velocity of a point on the perimeter of the wheel after 10 hits be?*

Mirek tampered with something again.

The hand is a point mass which has a moment of inertia with respect to the center of the wheel mr^2 at the time of each impact. The angular momentum transferred from the hand to the wheel in the n-th hit (after the hit, the hand is at rest with respect to the point of impact) is $mr(w - v_n)$, where v_n is the velocity of the point of impact after the n-th hit. The angular momentum of the wheel after the n-th hit is therefore

$$L_n = mr(w - v_n) + L_{n-1}\,.$$

That gives a recurrent formula for the velocities of the point of impact based on the formula $L_n = I\omega_n$

$$Av_n = w - v_n + Av_{n-1}\,,$$

where we used the substitution $A = I/(mr^2)$.

Now, we will try to compute the first few terms of the progression. We get

$$v_1 = \frac{w}{1+A}\,,\quad v_2 = \frac{w}{1+A}\left(1 + \frac{A}{1+A}\right)\,,\quad v_3 = \frac{w}{1+A}\left(1 + \frac{A}{1+A} + \left(\frac{A}{1+A}\right)^2\right)\,,\quad \dots$$

The formula for v_n will clearly be

$$v_n = \frac{w}{1+A}\left(1 + \frac{A}{1+A} + \dots + \left(\frac{A}{1+A}\right)^{n-1}\right)\,.$$

Summing up a geometric series and simplifying, we obtain

$$v_n = w\left(1 - \left(\frac{A}{1+A}\right)^n\right)\,.$$

The velocity is $v_{10} \doteq 4.54\,\mathrm{m{\cdot}s^{-1}}$.

Miroslav Hanzelka
mirek@fykos.cz

Problem FoL.49 ... all day long

Imagine that the observable universe is spherical with diameter $d = 10^{27}$ m and filled with stars placed in the lattice points of a cubic grid with side length of the unit cell $a = 10^{19}$ m. The stars are identical and their radiation is isotropic with luminosity $L = 10^{27}$ W and radius $r = 10^{9}$ m. The stars don't block other stars' light; there is no absorption. What's the incident power per square metre near the centre of the universe in the middle of a body diagonal of a unit cell? Compute the base-10 logarithm of this value! *Mirek was trying to see through a dense forest.*

A star's radiance at its surface is $\mathcal{L} = L/(4\pi r^2)$. The observer's distance from the stars in the corners of his unit cell is $a\sqrt{3}/2$. When looking at higher layers, i.e. cubes with side lengths $3a$, $5a$, $7a$ etc., centered at the observer's location, we may notice that the number of stars on each layer increases approx. quadratically with distance, since the number of stars is proportional to the area of the layer. At the same time, we know that a star's radiance decreases quadratically with distance, generally

$$L(r') = \frac{r^2}{r'^2} L(r)\,.$$

Introducing a density of stars $n = 1/a^3$, we're able to express the incident power as an integral

$$\mathcal{L}_{\mathrm{tot}} = \int_0^{d/2} \int_0^{2\pi} \int_0^{\pi} \mathcal{L} \frac{r^2}{\varrho^2} n \sin\vartheta \varrho^2 \,\mathrm{d}\vartheta \,\mathrm{d}\varphi \,\mathrm{d}\varrho = 4\pi \mathcal{L} r^2 \frac{d}{2} = \frac{d}{2a^3} L \doteq 5 \cdot 10^{-4}\,\mathrm{W{\cdot}m^{-2}}\,.$$

This result, however, could have a large error due to the contribution from nearby stars being inaccurately approximated – but we see that even the closest stars are far enough. The contribution from the nearest eight stars is

$$\mathcal{L}_8 = 8 \left(\frac{r}{a\sqrt{3}/2}\right)^2 \mathcal{L} \doteq 8 \cdot 10^{-12}\,\mathrm{W{\cdot}m^{-2}}\,,$$

which is completely negligible compared to $\mathcal{L}_{\mathrm{tot}}$ (we just need to realise that if the number of stars in a layer increases quadratically with the layer's dimensions and the radiance decreases quadratically as well, each layer must contribute approximately equally. It's possible to verify numerically that the approximation is accurate enough.[2]

Miroslav Hanzelka
`mirek@fykos.cz`

Problem FoL.50 ... comfy by the pond

It's summer and we're resting on a bank of a deep pond. The sun is shining, the acceleration due to gravity is $g = 9.81\,\mathrm{m{\cdot}s^{-2}}$, the surface tension of water is $\sigma = 72.8\,\mathrm{mN{\cdot}m^{-1}}$, water density is $\varrho = 1\,000\,\mathrm{kg{\cdot}m^{-3}}$. At a distance $d = 30.0$ m from the bank, a backswimmer appears and creates circular waves on the surface. When we focus on one wave, we can see that it's moving with speed $c = 1.00\,\mathrm{m{\cdot}s^{-1}}$. How long will it take for the wavefront of the backswimmer's waves to reach the bank? You may use the formula for phase velocity of deep-water waves $c = \sqrt{\frac{g}{k} + \frac{\sigma k}{\varrho}}$, where k is the wave number. *Mirek confused gravity waves for gravitational waves.*

[2] For the given values, it's not easy to perform the computation in reasonable time; it's better to test this assumption e.g. for $d = 10^3 a$.

The phase velocity is defined as the ratio of angular frequency and wave number,

$$c = \frac{\omega}{k}\,.$$

From this formula, we can express

$$\omega(k) = \sqrt{gk + \frac{\sigma k^3}{\varrho}}\,.$$

In order to determine how long it takes for the wavefront to reach us, we need to know the group velocity

$$c_\mathrm{g} = \frac{\partial \omega}{\partial k} = \frac{g + \frac{3\sigma k^2}{\varrho}}{2\sqrt{gk + \frac{\sigma k^3}{\varrho}}}\,.$$

The wave number k can be determined from the formula for phase velocity

$$c^2 = \frac{g}{k} + \frac{\sigma k}{\varrho}\,,$$

$$0 = k^2 - \frac{\varrho c^2}{\sigma} k + \frac{\varrho g}{\sigma}\,.$$

There's no need to substitute for k generally in the formula for group velocity. It's sufficient to solve the quadratic equation and substitute the right root (the one with minus – for small σ, k shouldn't approach infinity) into the expression for group velocity numerically. The group velocity can be plugged into the formula for the sought for time $t = d/c_\mathrm{g}$ to get

$$t = d\left(c - \frac{c}{2}\sqrt{1 - \frac{4\sigma g}{\varrho c^4}}\right)^{-1} \doteq 59.9\,\mathrm{s}\,.$$

Note that this result differs from the value $t = 60\,\mathrm{s}$, obtained by neglecting surface tension. In addition, it's very likely that the waves in a real pond would be damped before reaching the bank.

Miroslav Hanzelka
mirek@fykos.cz

Problem FoL.51 ... toast

We'd like to make a toast using a very strange glass. It's a hollow cube of inner volume $V = 1\,\mathrm{l}$, which is standing on one corner (a body diagonal is vertical). Through a small opening near the top vertex, we fill the cube with $V' = 1/3\,\mathrm{l}$ of wine. How high above the ground (i.e. the vertex the cube is standing on) will the wine surface be? Neglect the thickness of the cube's sides. Compute the result as a multiple of the cube's edge length!

Mirek remembered stereometry and had mixed feelings.

In order to solve the problem, it will be useful to first derive some formulae in a regular three-sided pyramid with right angles between pairs of edges at the main vertex. Let an edge of the base have length b, an edge at the main vertex have length a and its height be v. The relation between b and a is found easily from the Pythagorean theorem on one side of the pyramid to be

$$b(a) = \sqrt{2}\, a\,.$$

Next, we'll use the height of the base v_b, which is again found using the Pythagorean theorem for the triangle (base vertex)-(base vertex)-(foot of a height of the base) to be

$$v_\mathrm{b} = \frac{\sqrt{3}}{2} b\,.$$

The next is an expression for v using a and b. Let's denote the distance of a base vertex from the barycenter of the base by s. The barycenter splits each median into two parts with length ratio $2:1$. In addition, for a equilateral triangle (like our base), medians and heights coincide, so the barycenter and orthocenter also coincide. The distance s is

$$s = \frac{2}{3} v_\mathrm{b} = \frac{\sqrt{3}}{3} b\,.$$

Let's realise that in our pyramid, the foot of the height coincides with the base's orthocenter. Now, if we consider the Pythagorean theorem for the triangle (base vertex)-(main vertex)-(orthocenter), we get a formula for the height v of the pyramid:

$$v = \sqrt{a^2 - s^2} = \sqrt{\frac{1}{2} b^2 - \frac{1}{3} b^2} = \frac{\sqrt{6}}{6} b\,.$$

We'll also need the area S of the base, which is

$$S = \frac{1}{2} b v_\mathrm{b} = \frac{\sqrt{3}}{4} b\,.$$

Now, we can finally compute the volume V of the pyramid:

$$V = \frac{1}{3} v S = \frac{1}{3} \frac{\sqrt{3}}{4} \frac{\sqrt{6}}{6} b^3 = \frac{\sqrt{2}}{24} b^3\,,$$

but an expression using a will be more useful here:

$$V = \frac{1}{6} a^3$$

...and in v, it's

$$V(v) = \frac{\sqrt{3}}{2} v^3\,.$$

Let's return to our problem. If the volume of the wine inside the cube was at most $16\,\mathrm{l}^3 1\,\mathrm{l} = a^3$ the wine would be shaped like the above described pyramid, so we'd be pretty much done. The same goes for volume greater than $56\,\mathrm{l}$, where it's sufficient to compute the height of the empty part and subtract it from the body diagonal length $v_\mathrm{T} = \sqrt{3}\, a$. The critical volume is $16\,\mathrm{l}$ exactly because that's the volume of a pyramid obtained if we cut off the cube at its three vertices with smallest non-zero height.

[3],

Unfortunately, our volume really is in the range $\left\langle \frac{1}{6}, \frac{5}{6} \right\rangle$. Let's extend the edges of the cube from the corner on the ground upwards. Now, let's consider the plane parallel to the ground and at distance v from the ground, where $\frac{\sqrt{3}}{3}a \leq v \leq \frac{2\sqrt{3}}{3}a$, i.e. at other heights than in the cases described above. The corner on the ground and the intersection points of the extended edges are vertices of a regular three-sided pyramid, whose edges meet at right angles at the main vertex and whose height is v. However, this pyramid reaches out of the cube. Let's consider the three vertices of the cube that lie at the same height below the plane, i.e. just those three vertices which lie on one extended edge each. Let's consider the intersection points of edges of the cube that lead upwards from these vertices (6 points in total), in pairs respectively for each vertex. Each of these three points, together with its two intersection points and the intersection point of the extended edge with the plane, forms a pyramid. This pyramid has the same shape as above, its main vertex is the corresponding vertex of the cube and its height is $v - \frac{\sqrt{3}}{3}a$, because $\frac{\sqrt{3}}{3}a$ is the height of this corner of the cube above the ground. Those three pyramids fill the whole volume of the original large pyramid that extends beyond the cube. If we want to find the volume that's filled with the surface of wine at height v, it will be

$$V_{\mathrm{clk}}(v) = V(v) - 3V\left(v - \frac{\sqrt{3}}{3}a\right).$$

If we perform homogenisation $w = v/a$ and substitute for $V(v)$, we get

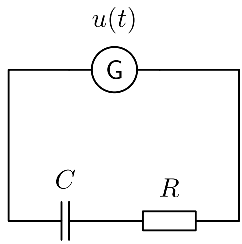
$$V_{\mathrm{clk}}(w) = \frac{\sqrt{3}}{2}\left(w^3 - 3\left(w - \frac{\sqrt{3}}{3}\right)^3\right).$$

You can test that by substituting $w = \frac{\sqrt{3}}{2}$, i.e. half the body diagonal, we get $V_{\mathrm{clk}} = \frac{1}{2}$, and by substituting critical heights $\frac{\sqrt{3}}{3}$ and $\frac{2\sqrt{3}}{3}$, we get the correct critical volumes $\frac{1}{6}$ and $\frac{5}{6}$, respectively. Now, we just need to set $V_{\mathrm{clk}} = \frac{1}{3}$ and solve the equation. This is a cubic equation with an analytical solution that can be expressed through the Cardano formulae, but in the contest, we only need the numerical result, which can be found efficiently e.g. using the service Wolfram Alpha; we need to pick the right root – the one that lies between the critical heights $\frac{\sqrt{3}}{3}$ and $\frac{2\sqrt{3}}{3}$. Numerically, this height is at 0.7347 of the cube's edge length.

Miroslav Hanzelka
mirek@fykos.cz

Problem FoL.52 ... saw

We connect a tone generator producing a sawtooth wave voltage $u(t)$ into a series RC circuit. The resistor in the circuit has resistance $R = 200\,\Omega$, the capacitor has capacity $C = 50\,\mu\mathrm{F}$, the maximum voltage of the source is $U = 5\,\mathrm{V}$. For the period of voltage $T = RC$, calculate the current passing through the circuit when the voltage of the source is at a maximum (specifically, instantly after the source voltage reaches the maximum) a long time after the generator is switched on.



Kuba thought there weren't any decent circuit problems yet.

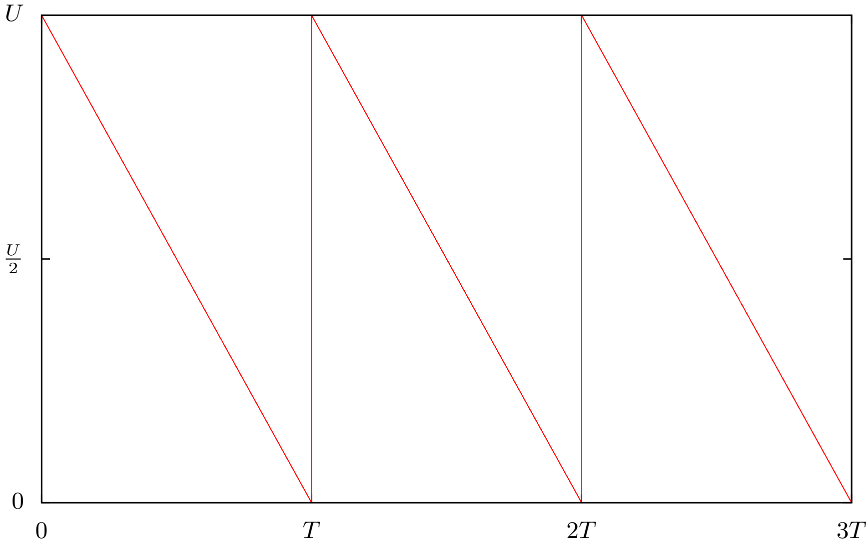


Fig. 2: Sawtooth wave voltage.

During one period, the voltage is

$$u(t) = U\left(1 - \frac{t}{T}\right)$$

and at the end, it discontinuously jumps back to $u(T) = U$.

Let's write the 2nd Kirchhoff law for this circuit during one period,

$$u(t) = RI(t) + \frac{q(t)}{C}\,, \tag{1}$$

where $I(t)$ is the current passing through the circuit at a time $t \in (0, T)$ and $q(t)$ is the charge of the capacitor. The polarity of this change is chosen so that it fits the equation (1) and we have

$$\dot{q}(t) = I(t)\,.$$

Differentiating (1) and substituting for $\dot{u}(t)$ and $\dot{q}(t)$, we obtain a differential equation

$$-\frac{U}{T} = R\dot{I}(t) + \frac{I(t)}{C}\,,$$

the solution of which is the current evolution in the form

$$I(t) = I_0 \exp\left(-\frac{t}{RC}\right) - \frac{CU}{T}\,, \tag{2}$$

with an unknown constant I_0, which we need to express.

Since the circuit is in stationary (periodically changing) state, the charge of the capacitor has to remain the same at the beginning and end of each cycle. The only difficulty is in thinking through what happens to the charge when voltage changes discontinuously. The change in charge, expressed as a current integral, will be zero in that (infinitely short) time interval, since the change in the current will be finite. Therefore, the net change in charge during the phase when voltage decreases has to be zero as well.

We may express the charge from (1), where we substitute for $I(t)$. We get

$$q(t) = CU\left(1 - \frac{t}{T}\right) + \frac{RC^2U}{T} - RCI_0 \exp\left(-\frac{t}{RC}\right) .$$

The zero change in charge between the beginning and end of the cycle is

$$0 = \Delta q = q(T) - q(0) = -CU + RCI_0\left[1 - \exp\left(-\frac{T}{RC}\right)\right] .$$

From that, we get

$$I_0 = \frac{U}{R\left[1 - \exp\left(-\frac{T}{RC}\right)\right]} .$$

Now, we may express the required current – in the equation (2), we substitute for I_0, set $t = 0$ and substitute the period $T = RC$ from the problem statement.

$$I(0) = \frac{U}{R\left[1 - \exp\left(-\frac{T}{RC}\right)\right]} - \frac{CU}{T} = \frac{U}{R}\frac{1}{\mathrm{e} - 1} \doteq 14.5\,\mathrm{mA} .$$

The current passing through the circuit in the given moments is 14.5 mA.

Jakub Dolejší
krasnykuba@fykos.cz

Problem FoL.53 ... you shall not transmit!

Consider a simple model of a static one-dimensional atmosphere, denoting its relevant coordinate (elevation) by h. If we took a thin slice of this atmosphere with thickness Δh, its reflectance would be $r(h)\Delta h$. What fraction of incident light is transmitted by the atmosphere if $r(h) = r_0\mathrm{e}^{-kh}$, where $k = r_0 = 0.1\,\mathrm{m}^{-1}$? *Kuba, an old organiser, decided to make a problem.*

First of all, let's consider an atmosphere of finite length L, with radiant flux I_0 perpendicularly incident on it from one side. Let's denote the coordinate normal to the atmospheric layer by x, so $x = 0$ on the side where the flux I_0 is incident and $x = L$ on the other side. Inside a layer, there are photons moving in both directions. Let's denote by $I_+(x)$ the flux in the direction of increasing x and by I_- the flux in the other direction. Then, it's clear that $I_+(0) = I_0$, $I_-(0) = RI_0$, $I_+(L) = TI_0$ and $I_-(L) = 0$, where R is the reflectance and T the transmittance of the atmosphere.

Considering a thin layer of thickness Δx within this atmosphere, we can write

$$\begin{aligned} I_+(x + \Delta x) &= \big(1 - r(x)\Delta x\big)I_+(x) + r(x)\Delta x I_-(x + \Delta x) , \\ I_-(x) &= \big(1 - r(x)\Delta x\big)I_-(x + \Delta x) + r(x)\Delta x I_+(x) , \end{aligned}$$

from which we express $I_+(x + \Delta x)$ and $I_-(x + \Delta x)$,

$$\begin{aligned} I_+(x + \Delta x)\big(1 - r(x)\Delta x\big) &= \big(1 - 2r(x)\Delta x\big)I_+(x) + r(x)\Delta x I_-(x) , \\ I_-(x + \Delta x)\big(1 - r(x)\Delta x\big) &= I_-(x) - r(x)\Delta x I_+(x) . \end{aligned}$$

If we expand this to the first order in Δx, we get in the limit $\Delta x \to 0$

$$\frac{\mathrm{d}I_+}{\mathrm{d}x} = r(x)(I_- - I_+)\,,$$
$$\frac{\mathrm{d}I_-}{\mathrm{d}x} = r(x)(I_- - I_+)\,.$$

This means $\mathrm{d}(I_- - I_+)/\mathrm{d}x = 0$, so $I_-(x) - I_+(x) = \mathrm{const} = -I_+(L) = -I_0 T$ and after substituting in the previous formula, $\mathrm{d}I_+/\mathrm{d}x = r(x)(I_- - I_+) = -r(x)I_0 T$, from which we may derive

$$-I_0 T \int_0^L r(x)\,\mathrm{d}x = I_+(L) - I_+(0) = (T-1)I_0\,.$$

From this, we can compute

$$T = \left(1 + \int_0^L r(x)\,\mathrm{d}x\right)^{-1}\,.$$

In our case, $r(x) = r_0 \mathrm{e}^{-k(L-x)}$ and $L \to \infty$, so

$$T_{\mathrm{atm}} = \lim_{L\to\infty} \left(1 + r_0 \mathrm{e}^{-kL} \int_0^L \mathrm{e}^{kx}\,\mathrm{d}x\right)^{-1} = \frac{k}{r_0 + k} = \frac{1}{2}\,.$$

The atmosphere will transmit 50 % of incident light.

Kuba Vošmera
kuba@fykos.cz

Problem FoL.54 ... transit

Consider a system in which infinitely many discrete states exist, indexed by an index $i \in \mathbb{N}$ (positive integers). The state of the whole system can be described by a state vector $\boldsymbol{f} = (f_1, f_2, \dots)$, where f_i denotes the number of objects in state i. The time evolution of the system occurs in discrete steps. The probability of transition of an object from a state i to the state $i+1$ in one step is $P_{i,i+1} = 1/i^2$, the probability of transition from a state i to the state $i-1$ is $P_{i,i-1} = 1 - P_{i,i+1}$. It follows from basic properties of probability that transitions between non-adjacent states or remaining in the same state isn't allowed. Initially, all objects are in state f_1. Find the stationary state the system converges to and compute the number of objects in the first state as a multiple of the total number of objects in the system N. Assume that N is a very large number. The total number of objects remains constant during the time evolution. *Mirek was transitioning from the dorm to school.*

The time evolution of the system can be described by a recurrence formula

$$\boldsymbol{f}_{t+1} - \boldsymbol{f}_t = A\boldsymbol{f}_t$$

which actually represents an infinite system of linear difference equations with coefficients determined by the matrix A. That matrix looks like this:

$$A = \begin{pmatrix} -1 & P_{21} & 0 & 0 & \cdots \\ P_{12} & -1 & P_{32} & 0 & \cdots \\ 0 & P_{23} & -1 & P_{43} & \cdots \\ 0 & 0 & P_{34} & -1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} -1 & 3/4 & 0 & 0 & \cdots \\ 1 & -1 & 8/9 & 0 & \cdots \\ 0 & 1/4 & -1 & 15/16 & \cdots \\ 0 & 0 & 1/9 & -1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

We can express the solution of this equation as a matrix exponential, but expressing it explicitly would require solving an infinite-dimensional eigenvalue problem. We can avoid that by using the simple form of the matrix and "seeing" the solution.

Let us denote the stable vector $\boldsymbol{f}^0$. W.l.o.g., let's take its first component to be $f_1^0 = 1$ (we can re-normalise it later). For this vector to remain constant when acted on by the matrix A, the number of objects to move from the second state to the first one must be 1, since objects can't move to the first state from any other. Thus, we necessarily have $f_2^0 = 4/3$, because $P_{21} = 3/4$. At the same time, we know that the number of objects that move to the second state from the first one is $P_{12} f_1^0 = 1$, so the number of objects that have to move from the third state to the second one is $4/3 - 1 = 1/3$. We know P_{32}, so it's easy to compute $f_3^0 = 3/8$. Using this method, we can compute the other components

$$f_4^0 = 2/45\,,\; f_5^0 = 5/1728\,,\; f_6^0 = 1/8400\,,\; \dots\,.$$

Generally, the component f_n^0 has to represent $1/(1-1/n^2)$ of the term that represents $1/(n-1)^2$ of the previous component f_{n-1}^0 (this can be derived by comparing the number of objects that pass in each direction between states n and $n-1$). We obtain a recurrence formula

$$f_n^0 = \frac{n^2}{n^2-1}\frac{1}{(n-1)^2} f_{n-1}^0\,.$$

This formula can be written using just the first element of the stable state vector $f_1^0 = 1$ (for $n > 1$)

$$f_n^0 = n^2 \prod_{k=0}^{n-2} \frac{1}{(n-k)^2 - 1}\,.$$

The total number of objects in the system can be expressed as

$$N = \sum_{n=0}^{\infty} f_n^0 = 1 + \sum_{n=0}^{\infty} n^2 \prod_{k=0}^{n-2} \frac{1}{(n-k)^2 - 1}\,.$$

Since this series converges quickly, it suffices to sum up the first few terms to get a fairly accurate result. Computer algebra systems tell us that the exact result is $N = 4 I_2(2) f_1^0 \doteq 2.756$, where I_j is the modified Bessel function of the j-th order. The problem asks for the inverse of this number, which is $1/(4 I_2(2)) \doteq 0.3629$.

Miroslav Hanzelka
mirek@fykos.cz

Problem FoL.55 ... megamagnet

There's a cylindrical coil of length $l = 100\,\mathrm{km}$ and base diameter $d = 1\,\mathrm{km}$ with $N = 10^6$ turns. There's a current $I = 1\,\mathrm{kA}$ passing through the coil. Determine the electric field felt when flying out of the coil parallel to its axis at a distance $r = 1\,\mathrm{m}$ from that axis with velocity $v = 1{,}000\,\mathrm{km{\cdot}s^{-1}}$. Neglect any effects of general relativity. *Xellos was watching Red Dwarf.*

Since $v \ll c$, we can neglect special relativity as well. We can also neglect Maxwell's current in the Ampere law and assume that the magnetic field sensed when moving is the same as at rest (if you don't agree, you can use the relativistic transform of the EM field).

The problem can now be solved using Maxwell's equations – specifically, the electric field has to satisfy Gauss's law div $\boldsymbol{E} = 0$ (there are no free charges) and Faraday's law of induction

$$\mathrm{rot}\,\boldsymbol{E} = -\frac{\mathrm{d}\boldsymbol{B}}{\mathrm{d}t} = -v\frac{\mathrm{d}\boldsymbol{B}}{\mathrm{d}z}$$

where we replaced the time changes of mag. field by movement along the axis. The magnetic field inside a long coil is almost parallel to the axis; at the end of the coil, it's starting to diverge from the axis, but only negligibly for $r \ll d$, so let's say that $\boldsymbol{B} = (0, 0, B_z(z))$ (we work in cylindrical coordinates (r, φ, z)).

The described system of equations is still too complex, especially since the el. field has 3 unknown components whose derivatives mix quite a lot. However, nothing stops us from guessing $E_z = 0$ (indeed, the field in the direction parallel to $\boldsymbol{v}$ doesn't change at all in the exact relativistic transform). Now, axial symmetry + Gauss's law give $E_r = 0$, since both E_φ and E_r only depend on r. Faraday's law simplifies (in the integral form) to $2\pi r E_\varphi = v\frac{\mathrm{d}B_z}{\mathrm{d}z}\pi r^2$ – imagine a coil with radius r sliding on the axis of the coil, then the left hand side gives the induced voltage in the coil and the right hand side the time change of mag. flux.

We still have to find $\frac{\mathrm{d}B_z}{\mathrm{d}z}$ near the axis at one end of the coil. "Near the axis" is important, because we know that the field near the axis for one coil centered at $(r = z = 0)$ with current I is given by Biot-Savart's law:

$$B_z(z) = \frac{\mu_0 I (d/2)^2}{2(z^2 + d^2/4)^{3/2}}\,.$$

We can imagine the coil to be infinite at the other end – it should have negligible effect on the mag. field – and imagine it to be composed of $\frac{xN}{l}$ thin current loops per x metres of length. The mag. field can be expressed as an integral over these loops:

$$B_z = \int_0^\infty \frac{\mu_0 I d^2}{(4z^2 + d^2)^{3/2}}\frac{N}{l}\mathrm{d}z\,.$$

We don't have to solve the integral (but we could, it gives $\mu_0 NI/2l$), since we're looking for its derivative by z – and that's simply equal to the integrand for $z = 0$. The resulting el. field is

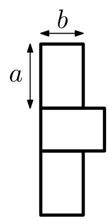
$$E = \frac{\mu_0 N I v r}{2ld} \doteq 6.3\,\mathrm{V{\cdot}m^{-1}}\,.$$

We can see that E is proportional to both v and r, which fits the expectations that $E = 0$ when $v = 0$ or $r = 0$.

Jakub Šafin
xellos@fykos.cz

Problem M.1 ... tower builder

We have glued together a tower from three identical blocks with edge lengths a, b (see the picture). Find the largest possible ratio a/b for which the tower would still be stable. *Náry and Lego.*



The condition for stability is that the centre of mass must lie above the base of the tower. To maximise a for given b, we want to put the centre of mass to its rightmost position – and that is b because there the base of the lower block ends.

Since the centre of mass of the lower and the upper block is at $b/2$ from the left edge, physics of the lever tells us that the center of mass of the middle block must be at $2b$. So the length of the block turns out to be $4b$ and the ratio in the question is $4 : 1$.

For completeness, the equation for the lever system is

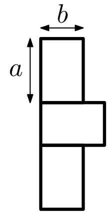
$$2Mg\left(x-\frac{b}{2}\right)=3Mg(x-b)\,,$$

where Mg is the weight of one block and x is the distance of middle block's center of mass from the left edge.

Jiří Nárožný
`nahry@fykos.cz`

Problem M.2 ... tower builder returns

We have built a tower from three identical blocks with edge lengths a, b. Find the largest possible ratio a/b for which the whole tower would not collapse immediately.
Náry and Lego.



We have to be more careful than in the previous task because we cannot use the argument about the center of gravity and the base of the tower which said that the stability is ensured if and only if the center is above the base. The blocks here can affect each other only by pressure, they can not pull each other, some of the forces and moments of force between them may not add to zero.

We shall inspect stability for each of the blocks separately. Stability of the bottom block is ensured automatically because its pressure to the ground is always compensated by the ground. The uppermost block is influenced by gravitation and pressure from the middle block, which is affected by both bottom and uppermost blocks together with gravitation. Let Mg denote the gravitational force acting on the uppermost block. The forces and torques (with respect to any point) must add to zero. The action and reaction principle says that a force with the same magnitude acts on the middle block at a point $b/2$ from its left side, where b is the width of the block.

Moreover, the middle block creates a force on the bottom one. This force is equal to the sum of gravitational forces of the uppermost and middle blocks, which are held up by the bottom one. Hence, the force is $2Mg$. In order to maximise the length of the edge a, we must maximise the distance from the center of gravity of the middle block to the place where the uppermost block acts. This is ensured if the bottom block acts on the one above it at its edge at distance b from the left side. We can use the law of the lever which leads us to conclude that the center of gravity of the middle block is $3b/2$ from its left side. Hence, the full length of the block is $3b$. The ratio is $3 : 1$.

For completeness, the condition for the lever (the law of the lever) has the form

$$2Mg(x-b)=Mg\left(x-\frac{b}{2}\right)$$

where x is the distance from the center of gravity of the middle block to its left side.

Jiří Nárožný
`nahry@fykos.cz`

Problem M.3 ... collision imminent

Calculate the lowest possible energy loss resulting from a perfectly inelastic central collision of two rigid balls with masses $m = 3\,\mathrm{kg}$ and $M = 2\,\mathrm{kg}$. The speeds before the collision are $5\,\mathrm{m\cdot s^{-1}}$ and $6\,\mathrm{m\cdot s^{-1}}$ (with respect to a static observer). *Náry, life experience.*

First we should realize that the task of minimizing the energy loss is equivalent to the task of maximizing the speed after the collision. Clearly, this can be achieved by sending both balls in the same direction.

Because the change of total kinetic energy with time is invariant under Galilean transform, it doesn't matter whether the heavier ball is the faster one or the slower one. The invariance will become obvious after the following calculation.

Let's denote v_1 the speed of the ball with mass m, v_2 the speed of the ball with mass M and w the final velocity. Conservation law for linear momentum implies

$$mv_1 + Mv_2 = (m + M)w\,,$$

and thus

$$w = \frac{mv_1 + Mv_2}{m + M}\,.$$

The decrease in total kinetic energy ΔE_k is given as the difference between total kinetic energy before and after the collision, that is

$$\frac{1}{2}mv_1^2 + \frac{1}{2}Mv_2^2 = \Delta E_\mathrm{k} + \frac{1}{2}(m + M)\left(\frac{mv_1 + Mv_2}{m + M}\right)^2\,.$$

Isolating ΔE_k we get

$$\Delta E_\mathrm{k} = \frac{1}{2}\frac{mM}{m + M}(v_1 - v_2)^2\,.$$

Now we can see that the energy loss is equal to the square of the velocity of one ball relative to the other. Therefore we can choose arbitrarily $v_1 = 5\,\mathrm{m\cdot s^{-1}}$, $v_2 = 6\,\mathrm{m\cdot s^{-1}}$. For given values of speeds and masses we get $\Delta E_\mathrm{k} \doteq 0.6\,\mathrm{J}$.

Jiří Nárožný
`nahry@fykos.cz`

Problem M.4 ... stick like no other

Rigid homogeneous stick of length $l = 1\,\mathrm{m}$ has been thrown upwards in a uniform gravitational field. During its flight, there was a moment when the translational kinetic energy equaled the rotational kinetic energy. We know that in that precise moment the speed of the stick's center of mass was $9\,\mathrm{m\cdot s^{-1}}$. Find the instantaneous angular velocity of the stick upon its release.
Everybody once met a rotating stick.

Since we are in a uniform gravitational field, the angular velocity, and by extension the rotational kinetic energy, are constant. When both components of the kinetic energy become equal, we can write

$$\frac{1}{2}mv^2 = \frac{1}{2}J\omega^2\,.$$

Notation: m is the mass of the stick, v is the translational velocity, J is the moment of inertia $ml^2/12$, l is the length of the stick, ω is the angular velocity. Through simple algebraic manipulation we obtain

$$\omega = \frac{2\sqrt{3}\,v}{l}\,.$$

Now we plug in the numbers and get $\omega \doteq 31.2\,\mathrm{rad\cdot s^{-1}}$.

Jiří Nárožný
nahry@fykos.cz

Problem E.1 ... charged Fykosák

Electrically neutral Fykos bird has been endowed with 10^{12} electrons. It then flies into a homogeneous electric field with speed $v_0 = 3.6\,\mathrm{km\cdot h^{-1}}$ in a direction opposite to the direction of the field lines. Electric intensity of the field is $500\,\mathrm{V\cdot m^{-1}}$. What will be the speed v of the Fykos bird after $100\,\mathrm{m}$ long flight in the electric field? Consider a point-like bird with mass $m = 0.5\,\mathrm{kg}$. *Faleš has been watching birds at a bus stop...*

Because of the charge of the bird, an electric force acts on it. The force is $F = qE$ and it will accelerate the bird because its charge is negative and therefore the force points in the direction opposite to the field.

The force does work (s is the length of trajectory of the bird in the field)

$$W = Fs\,,$$

which can be used in the conservation of energy

$$E_{\mathrm{ki}} = E_{\mathrm{kf}} + W_{\mathrm{e}}\,.$$

Here E_{ki} is the initial and E_{kf} is the final kinetic energy, which are given as follows

$$E_{\mathrm{k}} = \frac{1}{2}mv^2\,.$$

We can substitute and evaluate

$$v^2 = v_0^2 + \frac{2sqE}{m}\,.$$

Therefore

$$v = \sqrt{\frac{2s\cdot 10^{12}\cdot 1.6\cdot 10^{-19}\cdot E}{m} + v_0^2}\,.$$

Numerically, the result is $v \doteq 1.016\,\mathrm{m\cdot s^{-1}}$.

Note that we used only the mass of the bird, because the mass of the electrons is lower by 20 orders of magnitude and can be neglected.

Aleš Flandera
flandera.ales@fykos.cz

Problem E.2 ... Náry's hoverboard

How much current would have to flow through a horizontal wire 1 m *long to make the wire levitate above the ground? Magnetic induction is* $5 \cdot 10^{-5}$ T *at the place where the wire is and points to the north in the horizontal plane. The wire has a mass* 50 g. *The direction of the wire is from west to east.* *Faleš and Náry playing with a compass.*

The magnetic force has to be equal to the gravitational force and has to point upwards. The expression for magnetic force is

$$F_\mathrm{m} = BIl \sin \alpha \,,$$

where the angle α is measured between the vector of magnetic induction and the wire. In our setting, the angle is $\alpha = \pi/2$. We can express the current from the equation

$$BIl = mg \,.$$

That is,

$$I = \frac{mg}{Bl} \,.$$

Using the numbers gives us $I \doteq 9.8\,\mathrm{kA}$.

Aleš Flandera
`flandera.ales@fykos.cz`

Problem E.3 ... astonished Náry

Náry was playing with electrical components and ended up with four peculiar things. He was told by others that they are a resistor with resistance $R = 10\,000\,\Omega$, *capacitor with capacity* $C = 1\,\mu\mathrm{F}$, *inductor with inductance* $L = 10\,\mathrm{H}$ *and alternating current source with voltage* $U = 230\,\mathrm{V}$ *and frequency* $f = 50\,\mathrm{Hz}$. *By some strange coincidence he managed to build a series circuit. Knowing what he is playing with, he immediately computed the absolute value of phase shift between current and voltage in the circuit. What number did he get?*

Faleš and Náry during experimental afternoon.

The phase shift φ can be simply computed using the phase diagram, where on the positive y-axis we put voltage on the inductor, on its negative half the voltage on the capacitor and the voltage on the resistor is on positive x-axis. This gives us

$$\mathrm{tg}\,\varphi = \frac{U_L - U_C}{U_R} = \frac{I\omega L - \frac{I}{\omega C}}{IR} \,.$$

We can plug in formula for the angular frequency $\omega = 2\pi f$ to get

$$\mathrm{tg}\,\varphi = \frac{2\pi f L - \frac{1}{2\pi f C}}{R} \,.$$

The phase shift is

$$\varphi = \mathrm{arctg}\,\frac{2\pi f L - \frac{1}{2\pi f C}}{R} \,.$$

It gives us numerically $\varphi \doteq -0.24^\circ$. The absolute value therefore is 0.24°

Aleš Flandera
`flandera.ales@fykos.cz`

Problem E.4 ... it's freezing

An arc lamp has resistivity $0.2\,\Omega$ *and it is connected to voltage* $U = 60\,\mathrm{V}$*. How much heat does it release during* $1\,\mathrm{min}$*? Give the result in* MJ. *Faleš shivered with cold.*

We can use the Ohm's law to get the current through the lamp. It is

$$I = \frac{U}{R} = 300\,\mathrm{A}\,.$$

The heat, which is being released, is the Joule heat. The formula for it's flow (or power) P is

$$P = UI = \frac{U^2}{R}\,.$$

We need to multiply it by $t = 60\,\mathrm{s}$ to get the production in one minute. The result is $E = Pt = 1.08\,\mathrm{MJ}$.

Aleš Flandera
flandera.ales@fykos.cz

Problem X.1 ... Odysea and Prometheus

There are two space-ships, Odysea and Prometheus. Both of them have rest length L_0 *and both are approaching the Earth, but from opposite direction. They meet in a point where Odysea has speed* $0.25c$ *and Prometheus has speed* $0.75c$*. How long do they appear to each other in the moment when they pass each other? Determine the length as a multiple of* L_0*.*

Faleš has been watching Stargate.

First, we have to compute the combined velocity of both space-ships. They add together relativistically:

$$v'_{\mathsf{P}} = \frac{v_{\mathsf{P}} - v_{\mathsf{O}}}{1 - \frac{v_{\mathsf{O}} v_{\mathsf{P}}}{c^2}}\,,$$

Here v_{O} is velocity of Odysea (its speed with negative sign as they are approaching from opposite directions) and v_{P} is velocity of Prometheus (its speed with positive sign). If we plug in the numbers we get the total speed $16/19c$. Prometheus is therefore seem from Odysea to be approaching with speed $16/19c$ (and vice versa).

Now we can easily use the Lorentz formula for length contraction:

$$L' = \frac{L_0}{\gamma} = L_0\sqrt{1 - \left(\frac{16}{19}\right)^2} = \frac{\sqrt{105}}{19}L_0 \doteq 0.539L_0\,.$$

The ships see each other as $0.539L_0$ long.

Aleš Flandera
flandera.ales@fykos.cz

Problem X.2 ... who's gonna survive

Let us have a beam of neutrons with kinetic energy $T = 10\,\mathrm{keV}$. Mean lifetime of a neutron is $\tau_\mathrm{n} = 925\,\mathrm{s}$ and its rest energy is $m_\mathrm{n}c^2 = 939.6\,\mathrm{MeV}$. What percentage of neutrons in the beam decays during passage through distance $l = 10\,\mathrm{m}$? *Faleš was watching Independence day.*

Kinetic energy is much smaller than the rest mass and therefore we can solve the problem non-relativistically. Time needed by a neutron to travel a distance l is

$$t = \frac{l}{v} = \frac{l}{\sqrt{\frac{2T}{m_\mathrm{n}}}} = \frac{l}{c}\sqrt{\frac{m_\mathrm{n}c^2}{2T}}\,.$$

This time has to be substituted into the decay law. The portion of decayed neutrons is

$$f_\mathrm{D} = \frac{N_0 - N}{N_0} = 1 - \mathrm{e}^{-\frac{t}{\tau_\mathrm{n}}} = 1 - \mathrm{e}^{-\frac{l}{\tau_\mathrm{n}c}\sqrt{\frac{m_\mathrm{n}c^2}{2T}}} \approx \frac{l}{\tau_\mathrm{n}c}\sqrt{\frac{m_\mathrm{n}c^2}{2T}}\,.$$

Where we have used the Taylor expansion of an exponential. Numerically, we get $7.8 \cdot 10^{-7}\,\%$.

Aleš Flandera
flandera.ales@fykos.cz

Problem X.3 ... the winter, again

Consider a block of ice with mass $m_\mathrm{i} = 5\,\mathrm{kg}$ and temperature $t_\mathrm{i} = -2\,\mathrm{°C}$. We place $m_\mathrm{Fe} = 3\,\mathrm{kg}$ block of iron heated to $t_\mathrm{Fe} = 1\,333\,\mathrm{°C}$ on the block of ice. Determine the resulting temperature of the system. The specific heat capacity of ice is $c_\mathrm{i} = 2.1\,\mathrm{kJ{\cdot}kg^{-1}{\cdot}K^{-1}}$, the heat of fusion of ice is $l_\mathrm{im} = 334\,\mathrm{kJ{\cdot}kg^{-1}}$, the specific heat capacity of water is $c_\mathrm{w} = 4.18\,\mathrm{kJ{\cdot}kg^{-1}{\cdot}K^{-1}}$, the latent heat of evaporation of water is $l_\mathrm{vw} = 2.26\,\mathrm{MJ{\cdot}kg^{-1}}$ and the specific heat capacity of iron is $c_\mathrm{Fe} = 473\,\mathrm{J{\cdot}kg^{-1}{\cdot}K^{-1}}$ *Faleš has seen ice for the first time this year.*

First, the iron block heats the ice to temperature $0\,\mathrm{°C}$ and then the ice starts to melt (if there is enough heat in the iron). Therefore, let us start with comparison of the two heats. The heat needed to change temperature of an object from initial temperature t_i to temperature t_f is

$$Q = mc(t_\mathrm{f} - t_\mathrm{i})\,.$$

Plugging in the numbers we find the two heats are (their moduli) $Q_\mathrm{i} = 21\,\mathrm{kJ}$ and $Q_\mathrm{Fe} \doteq 1.89\,\mathrm{MJ}$.

We can see the ice is heated up and starts to melt. The heat needed to melt the entire block can be computed from equation

$$Q_\mathrm{im} = m_\mathrm{i} l_\mathrm{im}\,,$$

where l_im is heat of fusion of ice and its value is $334\,\mathrm{kJ{\cdot}kg^{-1}}$. The heat needed to melt the entire block of ice is $Q_\mathrm{im} = 1.67\,\mathrm{MJ}$. Therefore the block really is melted and starts to heat up. The final equation therefore is

$$m_\mathrm{Fe}c_\mathrm{Fe}(t_\mathrm{Fe} - t) = Q_\mathrm{i} + Q_\mathrm{im} + m_\mathrm{i}c_\mathrm{w}t\,,$$

where $c_\mathrm{w} = 4.18\,\mathrm{kJ{\cdot}kg^{-1}{\cdot}K^{-1}}$ is specific heat capacity of water. The resulting temperature is

$$t = \frac{m_\mathrm{Fe}c_\mathrm{Fe}t_\mathrm{Fe} - (Q_\mathrm{i} + Q_\mathrm{im})}{m_\mathrm{i}c_\mathrm{w} + m_\mathrm{Fe}c_\mathrm{Fe}} = \frac{m_\mathrm{Fe}c_\mathrm{Fe}t_\mathrm{Fe} - m_\mathrm{i}c_\mathrm{i}\,(t_0 - t_\mathrm{i}) - m_\mathrm{i}l_\mathrm{im}}{m_\mathrm{i}c_\mathrm{w} + m_\mathrm{Fe}c_\mathrm{Fe}}\,.$$

Numerically, this evaluates to $t \doteq 9.0\,^\circ\mathrm{C}$.

Aleš Flandera
flandera.ales@fykos.cz

Problem X.4 ... tiler

Are there such plane edge-to-edge tilings where, in every vertex, t equilateral triangles, s squares, and h regular hexagons meet? We want to know how many such tilings exist. Numbers t, s, h are non-negative integers. Sizes of polygons of the same type in one tiling are identical. The arrangement of polygons about the vertex does not matter (different arrangements for fixed s, t, h count as one). *Mirek got a new book.*

For every such tiling

$$\frac{\pi}{3}t + \frac{\pi}{2}s + \frac{2\pi}{3}h = 2\pi\,,$$

must hold, that is

$$2t + 3s + 4h = 12\,.$$

Now we can see that

$$t \leq 6\,, s \leq 4\,, h \leq 3\,.$$

There are 7 possible combinations of (t, s, h): $(0,0,3)$, $(0,4,0)$, $(6,0,0)$, $(1,2,1)$, $(2,0,2)$, $(3,2,0)$, $(4,0,1)$. Of course, we can find diferent arrangements for these triplets (try $(1,2,1)$), but we asked only for the number of triplets, so the answer is 7.

Miroslav Hanzelka
mirek@fykos.cz

FYKOS
UK, Matematicko-fyzikální fakulta
Ústav teoretické fyziky
V Holešovičkách 2
180 00 Praha 8

www: http://fykos.cz
e-mail: fykos@fykos.cz

FYKOS is also on Facebook
http://www.facebook.com/Fykos

FYKOS is organized by students of Faculty of Mathematics and Physics of Charles University. It's part of Public Relations Office activities and is supported by Institute of Theoretical Physics, MFF UK, his employees and The Union of Czech Mathematicians and Physicists.