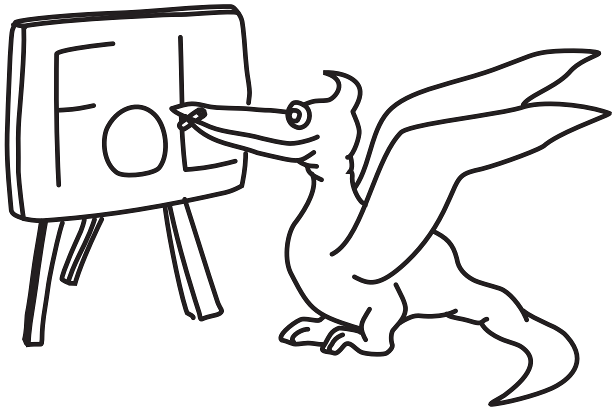
Solutions of 8th Online Physics Brawl



Problem FoL.1 ... underground 3 points

Matěj, with mass $m = 70\,\mathrm{kg}$, entered an underground train car and spotted his friend Jáchym at the other side of the train car, at distance $s = 20\,\mathrm{m}$. Matěj started walking towards Jáchym, in the direction of motion of the train. The train was moving with constant acceleration $a = 1\,\mathrm{m{\cdot}s^{-2}}$. How much work must Matěj do to reach Jáchym?

Matěj loves travelling by underground.

In a train car traveling with a constant velocity, the mechanical work done by Matěj would be zero, since his initial and final position are at the same height and there is no opposing force. While accelerating, the underground becomes a non-intertial reference frame. Matěj is heading in the direction of train car's acceleration, so he must do some work to actually move. Newton's second law says in this case that the force exerted by Matěj in the direction of his motion is (assuming he moves with a constant velocity)

$$F = ma\,.$$

This force is kept constant along the path of length s, so the work done by Matěj is

$$W = Fs = mas = 1{,}400\,\mathrm{J}\,.$$

This is equivalent to the energy contained in about 10 milligrams of sugar (approx. 50 grains). It seems Matěj won't strain too much.

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Problem FoL.2 ... passing trains 3 points

Danka was watching a train with length $l_1 = 120\,\mathrm{m}$, which was passing her with velocity $v_1 = 60\,\mathrm{km{\cdot}h^{-1}}$. She also saw a train of unknown length l_2 approaching from the opposite direction with velocity $v_2 = 80\,\mathrm{km{\cdot}h^{-1}}$. In order to find l_2, Danka measured the time the trains spent passing each other, i.e. the interval between the times when the fronts of the trains met and when the backs of the trains met. She measured $t = 9\,\mathrm{s}$. How long was the second train?

Danka was waiting for a train.

This problem is best solved in the reference frame of the first train. Then the second train approaches the first (stationary) train with velocity $v_1 + v_2$. In order for the trains to pass each other, the second train must travel the distance equal to the length of the first train (so that the front of the second train reaches the back of the first train) plus the length of the second train (so that the backs of both trains are next to each other). Let's denote the total travel time t and the distance covered

$$s = l_1 + l_2\,.$$

It follows

$$l_1 + l_2 = (v_1 + v_2)t\,.$$

Now we express the length l_2 as

$$l_2 = (v_1 + v_2)t - l_1 = 230\,\mathrm{m}\,.$$

The length of the second train is 230 m.

Daniela Pittnerová
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Problem FoL.3 ... let's play a game 3 points

You are playing a game which consists of separate rounds. When you lose a round, you lose a point. When you win a round, you gain a point, and if you have also won the previous two rounds, you gain another point as a bonus. If the chance of winning a round is 50 % (draw is not possible), what is the long-term average (expected value) of the number of points gained per round? *Jáchym plays only ranked matches in Hearthstone.*

Let's assume a round of the game just started. There's a 50 % chance we will lose this round and thus lose a point, and there's a 50 % chance we will win and gain a point. Independently of those probabilities we have won both previous rounds with probability 25 %. The probability to gain a bonus point is therefore 12.5 %. Putting all this together we get

$$p = 0.5 \cdot (-1) + 0.5 \cdot 1 + 0.125 \cdot 1 = 0.125 \,.$$

The expected point gain per round is 0.125.

Jáchym Bártík
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Problem FoL.4 ... Keep pulling or I'll whip you! 3 points

When building the pyramids, the transport of giant stone blocks was done by exploitation of the good old slave labor. The blocks are transported on an inclined plane with slope $\alpha = 15^\circ$. Whipmaster Jáchym found out that to pull a block up the inclined plane, the full force of 15 slaves is needed. However, when transporting the same block down, he only needs five slaves pulling with full power. Determine the friction coefficient between the block and the inclined plane. Each slave is equally strong and pulls a rope parallel to the inclined plane with constant force. *Matěj was interested in pulling of blocks.*

Let us denote the weight of a block by m and the friction coefficient by f. The normal force pushing the block to the plane is $mg\cos\alpha$. Then the friction force is given by $fmg\cos\alpha$. When the slaves pull a block downward, they are help by the tangential component of the gravitational force $mg\sin\alpha$. When they pull the same block upward, they have to compensate for the same force component. The maximum force exerted by one slave will be denoted F in the following. We get two balance equations (one for pulling upward, one for pulling downward)

$$15F = fmg\cos\alpha + mg\sin\alpha \,,$$
$$5F = fmg\cos\alpha - mg\sin\alpha \,.$$

Solving for the friction coefficient we obtain

$$fmg\cos\alpha + mg\sin\alpha = 3fmg\cos\alpha - 3mg\sin\alpha \,,$$
$$4\sin\alpha = 2f\cos\alpha \,,$$
$$f = 2\tan\alpha \,.$$

For given values we get $f \doteq 0.536$.

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Problem FoL.5 ... Theophist temperature

3 points

In hell, temperature is measured on the Theophist scale. The conversion formula between Theophist and Celsius scales is linear. The temperature $100\,^\circ\mathrm{T}$ *(degrees Theophist) is the boiling temperature of sulfur* $445\,^\circ\mathrm{C}$*. Once it was so cold in hell that paraffin wax solidified. At that time, the hell's meteorological station measured* $-62\,^\circ\mathrm{T}$*, which corresponds to* $40\,^\circ\mathrm{C}$*. Determine the boiling temperature of mercury* ($357\,^\circ\mathrm{C}$) *in hell's units* $^\circ\mathrm{T}$*.*

You are going to get into hot mercury about that.

Let us define the conversion formula $y(^\circ\mathrm{T}) = kx(^\circ\mathrm{C}) + q$ for the two temperature scales, where x is the temperature given in degrees of Celsius, y is the temperature given in degrees of Theophist and q and k are some coefficients. Then by solving the set of equations

$$100 = 445k + q\,,$$
$$-62 = 40k + q$$

we obtain $k \doteq 0.4$ a $q \doteq -78$. We substitute for the coefficients in the conversion formula $y(^\circ\mathrm{T}) = 0.4x(^\circ\mathrm{C}) - 78$. Plugging in the value for the boiling temperature of mercury in Celsius we get $65\,^\circ\mathrm{T}$.

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Problem FoL.6 ... stairs

3 points

Danka steps on the top stair of an escalator. When she is standing still, the escalator carries her to the bottom in $t_1 = 2.0\,\mathrm{min}$*. If the escalator was not moving and Danka walked down it, it would take her* $t_2 = 2.5\,\mathrm{min}$ *to get to the bottom. How long will it take her to get from the top to the bottom if she starts walking down the moving escalator in the middle of the path? Danka walks with constant velocity.*

Danka stopped to think in an underground.

Denote v_1 the velocity of the escalator, v_2 the velocity of Danka walking and t_3 the time in question. If we further denote l the length of the escalator, then we can write for the given velocities

$$v_1 = \frac{l}{t_1}\,,$$
$$v_2 = \frac{l}{t_2}\,.$$

For half of the path Danka is standing still, so she covers this segment in $t_1/2$. Her velocity on the other half of the escalator is given by the sum of v_1 and v_2. The time of travel, t_3, is then expressed through

$$t_3 = \frac{t_1}{2} + \frac{\frac{l}{2}}{v_1 + v_2}\,,$$
$$t_3 = \frac{t_1}{2} + \frac{1}{2}\frac{l}{\frac{l}{t_1} + \frac{l}{t_2}}\,,$$
$$t_3 = \frac{t_1\left(t_1 + 2t_2\right)}{2\left(t_1 + t_2\right)} \doteq 1.6\,\mathrm{min}\,.$$

The travel on escalator will take Danka 1.6 min.

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Problem FoL.7 ... hockey-football problem 4 points

There is a radioactive substance with N_0 particles. Let's denote the number of particles that decay during the second $1/3$ of the first half-life by ΔN. Find the ratio $\Delta N/N_0$.

Waiting for the third third of Half-Life.

The number of undecayed particles after the first third of a half-time is

$$N_1 = N_0 \left(\frac{1}{2}\right)^{\frac{1}{3}} ,$$

after the second third

$$N_2 = N_0 \left(\frac{1}{2}\right)^{\frac{2}{3}} .$$

In result,

$$\frac{N_1 - N_2}{N_0} = \left(\frac{1}{2}\right)^{\frac{1}{3}} - \left(\frac{1}{2}\right)^{\frac{2}{3}} \doteq 0.1637$$

is the portion of particles that decayed during the second third of a half-time.

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Problem FoL.8 ... resistive block 3 points

We have $l = 11.4\,\mathrm{m}$ of wire with circular cross-section with diameter $d = 0.61\,\mathrm{mm}$. The resistance of the wire is $R = 19.6\,\Omega$. What would be the resistance R_a of a cube formed by smelting this wire, if it was measured under direct current, between two perfectly conductive plates touching two opposite sides of the cube? *Karel was thinking up simple physics problems.*

For resistance of a wire with circular cross-section with length l and cross-section $S = \pi d^2/4$, the following formula can be applied:

$$R = \varrho \frac{l}{S} = \varrho \frac{4l}{\pi d^2} ,$$

where ϱ is material resistivity of which the wire is made out, and which we want to obtain from the relation

$$\varrho = \frac{\pi d^2 R}{4l} .$$

We can work out the resistance of a cube as

$$R_a = \varrho \frac{l_a}{S_a} = \frac{\pi d^2 R}{4l} \frac{a}{a^2} = \frac{\pi d^2 R}{4la} ,$$

where a is the length of the edge of the cube, $l_a = a$ and $S_a = a^2$. Now we have to work out the length of the cube's edge. This can be determined from the conserved mass and volume

$$V_a = V \quad \Rightarrow \quad a^3 = lS \quad \Rightarrow \quad a = \sqrt[3]{\frac{\pi d^2 l}{4}} .$$

We substitute into the resistance formula and get

$$R_a = \frac{\pi d^2 R}{4l} \sqrt[3]{\frac{4}{\pi d^2 l}} = \left(\frac{\sqrt{\pi} d}{2l}\right)^{\frac{4}{3}} R \doteq 3.36 \cdot 10^{-5}\,\Omega\,.$$

The resistance of the cube with contacts on opposite sides would be only $3.36 \cdot 10^{-5}\,\Omega$. As a side note, the length of the cube is around 1.5 cm.

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Problem FoL.9 ... queen's justice 3 points

Poor serfs are bringing eggs to the queen and receive gold coins in exchange. Each egg is worth ten gold coins. However, if a serf brings eggs worth more than 6,700 *gold coins in total, the queen will pay* 1,600 *gold coins less for them to make sure landowners won't get as rich as Her Majesty. If they bring eggs worth from* 5,000 *to* 6,700 *gold coins (inclusive), the pay is reduced only by "only"* 1,000 *gold coins.*

A farmer is breeding hens that lay from 0 to 1,000 *eggs per week (uniformly randomly distributed), which he brings to the queen each Sunday. What is the average (expected) amount of gold coins he earns per week?* *Matěj and Jáchym are obedient servants of Her Majesty.*

Ignoring the money deduction the farmer would earn on average 5,000 gold coins each week (the expected value of the uniform distribution). In 50 % of all cases there will be no deduction because the farmer's weekly earnings will be less than 5,000 (actually the probability is a little bit lower since there are 1,001 possible cases and only in 500 of them there is no deduction, but we can neglect this). Then there is a 17% probability of facing a 1,000 tax and 33% probability of paying 1,600 in tax. The resulting average is

$$5{,}000 - 0.17 \cdot 1{,}000 - 0.33 \cdot 1{,}600 = 4{,}302$$

gold coins.

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Problem FoL.10 ... scuba diver 4 points

A scuba diver is $h = 10\,\mathrm{m}$ under the water surface. How large is the area of the surface on which he can see the sky? The refractive index of water is 1.33 *and the refractive index of air is* 1.00. *Šimon is reminiscing about summer.*

The refraction of light passing through a boundary between two media is described by the well-known Snell's law

$$n_1 \sin \alpha_1 = n_2 \sin \alpha_2\,,$$

where in this case n_1 denotes the refractive index of water, n_2 is the refractive index of air and α_1 stands for the angle of refraction. The maximum value of the angle of incidence α_2 is 90°. For this value we can find the critical angle of refraction

$$\sin \alpha_\mathrm{m} = \frac{1}{n_1}\,.$$

Invoking the symmetry properties of the given problem, we conclude that the surface area, where the sky reflection is visible, is a circle. The radius of this circle r can be expressed with the help of the tangent function

$$\tan \alpha_\mathrm{m} = \frac{r}{h}\,.$$

Then the area of the circle is given by

$$S = \pi r^2 = \pi h^2 \tan^2 \alpha_\mathrm{m} = \pi h^2 \frac{\sin^2 \alpha_\mathrm{m}}{\cos^2 \alpha_\mathrm{m}} = \pi h^2 \frac{\sin^2 \alpha_\mathrm{m}}{1 - \sin^2 \alpha_\mathrm{m}}\,,$$

where we expressed radius r in terms of depth of submersion of the scuba diver h and angle α_m. In the last two steps we also used trigonometric formulas to substitute tan with the sine function. The final result is

$$S = \pi h^2 \frac{1/n_1^2}{1 - 1/n_1^2} = \pi h^2 \frac{1}{n_1^2 - 1}\,,$$

which is approximately $409\,\mathrm{m}^2$.

Šimon Knoška
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Problem FoL.11 ... fifty

3 points

Vašek calibrated his tachometer so that it would show the exact velocity of his car based on the rotation speed of the wheels. However, his tyres are worn and their radius is $35.7\,\mathrm{cm}$, so he replaces them by new ones with $4\,\mathrm{mm}$ larger radius, but forgets to recalibrate the tachometer. How much (in $\mathrm{km{\cdot}h^{-1}}$) does his real velocity differ from the velocity displayed on the tachometer, which is $50\,\mathrm{km{\cdot}h^{-1}}$? *Vašek follows traffic laws.*

Car's velocity is measured via rotational speed of the wheels, so the velocity displayed on the tachometer is calculated from angular frequency ω of the wheels. The velocity v on the tachometer is equal to the velocity of the car with worn tyres of a radius R. After Vašek changes tyres from the old ones to the new with radius $R' = R + \Delta R$, he drives with a velocity $v' = \omega R'$, whereas (because angular frequency ω remains the same) tachometer still shows velocity $v = \omega R$. We want to find the difference Δv of these velocities,

$$\Delta v = v' - v = \omega \left(R' - R\right) = v \frac{\Delta R}{R}\,.$$

In conclusion, Vašek breaks the speed limit by $0.56\,\mathrm{km{\cdot}h^{-1}}$.

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Problem FoL.12 ... Watch out for the turn!

3 points

A car with mass $m = 1.50\,\mathrm{t}$ is taking a turn with radius $R = 30.0\,\mathrm{m}$. The static friction coefficient between the tyres and road surface is $f = 0.55$. With what maximum velocity may the car take the turn without skidding? *Danka took a turn too fast.*

Centrifugal force

$$F_\mathrm{c} = m\frac{v^2}{R}$$

acts in the outward direction on the car taking the turn. A static friction force acts in the opposite direction and compensates F_c completely. However, the strength of static friction is limited by

$$F_\mathrm{t,max} = fmg\,.$$

In order to avoid a car slide, the centrifugal force must be less than or equal to the maximum friction force

$$F_\mathrm{c} < F_\mathrm{t,max}\,,$$

$$m\frac{v^2}{R} < fmg\,.$$

The condition on allowed values of velocity follows

$$v < \sqrt{fgR} \doteq 45.8\,\mathrm{km{\cdot}h^{-1}}\,.$$

The car can take the turn with a maximum velocity of $45.8\,\mathrm{km{\cdot}h^{-1}}$.

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Problem FoL.13 ... Planck linear density 5 points

Planck units are physical quantities formed as suitable combinations of speed of light $c = 3.00\cdot 10^{8}\,\mathrm{m{\cdot}s^{-1}}$, reduced Planck constant $\hbar = 1.05\cdot 10^{-34}\,\mathrm{kg{\cdot}m^2{\cdot}s^{-1}}$ and the gravitational constant $G = 6.67\cdot 10^{-11}\,\mathrm{kg^{-1}{\cdot}m^3{\cdot}s^{-2}}$. That means we may write $A = c^\alpha\cdot\hbar^\beta\cdot G^\gamma$, obtaining all sorts of physical quantities. Consider such an expression for linear density, which is usually denoted by λ and has units $\mathrm{kg{\cdot}m^{-1}}$. Compute the sum of exponents $\alpha+\beta+\gamma$.

Karel likes to think about Planck's units.

Problems of this kind are usually solved through dimensional analysis, so let's try that. The assumption is

$$\lambda = c^\alpha\cdot\hbar^\beta\cdot G^\gamma\,.$$

Plugging in the physical units, this amounts to

$$\mathrm{kg{\cdot}m^{-1}} = \mathrm{m}^\alpha{\cdot}\mathrm{s}^{-\alpha}{\cdot}\mathrm{kg}^\beta{\cdot}\mathrm{m}^{2\beta}{\cdot}\mathrm{s}^{-\beta}{\cdot}\mathrm{kg}^{-\gamma}{\cdot}\mathrm{m}^{3\gamma}{\cdot}\mathrm{s}^{-2\gamma}\,.$$

For each unit (kg, m and s) we get a linear equation, resulting in a set

$$1 = 0\alpha + \beta - \gamma\,,$$
$$-1 = \alpha + 2\beta + 3\gamma\,,$$
$$0 = -\alpha - \beta - 2\gamma\,.$$

We can easily solve this and obtain $\alpha = 2$, $\beta = 0$ and $\gamma = -1$, and so

$$\lambda = \frac{c^2}{G} \doteq 1.35\cdot 10^{27}\,\mathrm{kg{\cdot}m^{-1}}\,.$$

An alternative approach starts with the definitions of Planck mass and Planck length, which can be then divided to immediately obtain

$$\lambda = \frac{m}{l} = \sqrt{\frac{c\hbar}{G}}\sqrt{\frac{c^3}{\hbar G}} = \frac{c^2}{G}\,.$$

The task requires us to sum the exponents we found, so the final result is 1.

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Problem FoL.14 ... drop velocity

3 points

Compute the velocity of a raindrop (a sphere with radius 3 mm) immediately before hitting the ground. Assume the flow of air around a raindrop is turbulent. The density of air is $\varrho = 1.25\,\mathrm{kg{\cdot}m^{-3}}$ and the drag coefficient of a spherical raindrop is $C = 0.5$.

Katarína likes walking in the rain.

When a rain droplet falls to the ground, gravitational force and drag force affect it. The gravitational force is given as $F_G = mg$. The drag force is (considering turbulent flow) $F_\mathrm{D} = \frac{1}{2}CS\varrho v^2$ and it depends on the drag coefficient C, which is a constant, the cross-sectional area of falling object S, the density of the fluid ϱ, and the velocity of the falling object v. During the fall, the object's velocity changes, so the drag force affecting it changes as well. When the drag force becomes equal to the gravitational force effecting the droplet, there is no resultant force effecting the object and thus the object continues to fall with its terminal velocity. We need to find this velocity from the equation

$$F_\mathrm{D} = F_G \qquad \Rightarrow \qquad \frac{1}{2}CS\varrho v^2 = mg\,.$$

Drag coefficient of a sphere is a constant with known value $C = 0.50$. Also, we know the cross-sectional area of the droplet, calculated from known variables as $S = \pi r^2$, where r is radius of the droplet. The density of the surroundings is the density of air since the droplet is surrounded by air. We want to find the velocity of the droplet, so v is our unknown. We can work out mass of the droplet from its density and volume:

$$m = \varrho_\mathrm{water} V_\mathrm{droplet} = \varrho_\mathrm{water}\frac{4}{3}\pi r^3\,.$$

After rearranging the first equation, we get the desired formula for the velocity of the droplet

$$v = \sqrt{\frac{\varrho_\mathrm{water}\frac{4}{3}2rg}{C\varrho_\mathrm{air}}} \doteq 11.2\,\mathrm{m{\cdot}s^{-1}}\,.$$

The velocity of the droplet when it hits the ground is $11.2\,\mathrm{m{\cdot}s^{-1}}$. We accentuate that this solution works only when there is enough time for gravitational and drag forces to reach an equilibrium.

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Problem FoL.15 ... giga-cave 5 points

Consider a perfectly spherical planet with radius R. The planet is homogeneous, except for a spherical cave with radius $R/8$ touching the surface of the planet. What is the ratio of gravitational acceleration close to the surface at the point where the cave touches the surface to gravitational acceleration at the opposite point? *Karel likes spherical planets.*

Gravitational interaction of individual physical objects is additive. Therefore, we may model our planet as the difference of a full homogeneous sphere and a small full sphere located in the cave. Let ϱ be the density of the material of the planet. Then the mass of the full sphere equals

$$M = \frac{4}{3}\pi R^3 \varrho \,,$$

whereas for the smaller full sphere we have

$$m = \frac{4}{3}\pi \left(\frac{R}{8}\right)^3 \varrho .$$

For gravitational acceleration on a surface of the planet directly above the cavity we obtain

$$g_1 = \frac{GM}{R^2} - \frac{Gm}{\left(\frac{R}{8}\right)^2} = \frac{7}{6}\pi R G \varrho \,,$$

and on the opposite side

$$g_2 = \frac{GM}{R^2} - \frac{Gm}{\left(\frac{15R}{8}\right)^2} = \frac{1{,}799}{1{,}350}\pi R G \varrho \,.$$

Final ratio of these accelerations is

$$\frac{g_1}{g_2} = \frac{225}{257} \,,$$

which roughly equals 0.875.

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Problem FoL.16 ... compound bow or semiautomatic pistol? 5 points

We fire an arrow from a compound bow PSE X-Force Omen and a bullet from a semi-automatic pistol Colt 1903 .32 directly against each other. If these two projectiles collided in a perfectly inelastic collision and merged together, what would be the velocity (including the direction) of the resulting projectile? The arrow has mass $m_1 = 350\,\mathrm{grain}$ (as you surely know, $1\,\mathrm{grain} = 6.479 \cdot 10^{-5}\,\mathrm{kg}$) and velocity $v_1 = 366\,\mathrm{ft{\cdot}s^{-1}}$ (feet per second, $1\,\mathrm{ft{\cdot}s^{-1}} = 0.3048\,\mathrm{m{\cdot}s^{-1}}$). The bullet, 7.65 mm Browning Short, has mass $m_2 = 65\,\mathrm{grain}$ and velocity $v_2 = 925\,\mathrm{ft{\cdot}s^{-1}}$. Assume non-rotating point projectiles. The answer should be positive if the resulting projectile flies in the direction of the arrow or negative if it flies in the direction of the bullet.

Karel was thinking about ranged weapons.

Total mechanical energy is not conserved in case of perfectly inelastic collision. On the other hand we can use conservation of linear momentum p and fact, that after the collision projectiles move together with a common velocity w. As a positive direction we choose direction of the arrow as mentioned in the task.

$$p = m_1 v_1 - m_2 v_2 = (m_1 + m_2) w\,.$$

We can express the velocity as

$$w = \frac{m_1 v_1 - m_2 v_2}{m_1 + m_2} \doteq 164\,\mathrm{ft{\cdot}s^{-1}} \doteq 50\,\mathrm{m{\cdot}s^{-1}}\,.$$

As we may see, it is better to stand on the side of the bowman. Arrow "pushes" the bullet and they continue with relatively high velocity (roughly 45 % of original velocity of the arrow) towards gunman.

Of course, the in-flight collision of arrow and bullet is not very realistic. Even if they collided, they would probably just change their directions or fracture. To conclude, we should mention that the original velocities in common units are $v_1 = 112\,\mathrm{m{\cdot}s^{-1}}$ and $v_2 = 282\,\mathrm{m{\cdot}s^{-1}}$ and kinetic energies of projectiles are $E_1 = 141\,\mathrm{J}$ and $E_2 = 167\,\mathrm{J}$.

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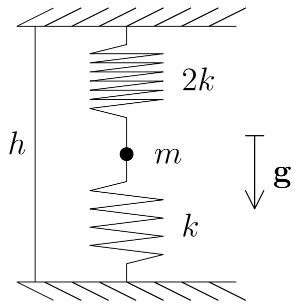
Problem FoL.17 ... twice as stiff spring 3 points



There are two springs with negligible masses and one point mass in the configuration depicted in the figure. The stiffness of the lower spring is $k = 5.0\,\mathrm{kg{\cdot}s^{-2}}$ and the stiffness of the upper spring is $2k$. The rest length of each spring is $l_0 = h/4 = 10\,\mathrm{cm}$ and they are both stretched to lengths $2l_0$. What should be the mass of the point mass if we want the whole configuration to be in equilibrium? The acceleration due to gravity is $g = 9.81\,\mathrm{m{\cdot}s^{-2}}$.

Karel varied Paťo's problem from Slovak Physics Olympiad.

We can divide the upper spring with stiffness $2k$ into two springs with the same rest length and with stiffness k. Force from one of them cancels force from the lower spring and so only one spring with stiffness k and a point of mass is left. Elongation of upper spring is $\Delta l = l_0$, so we have an equation

$$mg = \Delta l k = l_0 k\,,$$

from which we get

$$m = \frac{l_0 k}{g} \doteq 51\,\mathrm{g}\,.$$

The mass is 51 g.

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Problem FoL.18 ... a warm-up hurricane 4 points

The main driving force of a cyclone is the ocean, or more specifically, the heat lost by water evaporating from the surface of the ocean. Steam particles are quickly lifted up to the upper part of the cyclone, where water condensates due to lower temperature, pressure and presence of condensation nuclei. In the last part of the cycle, it falls down to the ocean surface as rain (this process is also very fast). Therefore, the cyclone works between two temperatures – the ocean temperature $T_1 = 280\,\mathrm{K}$ and the cloud temperature $T_2 = -26\,{}^\circ\mathrm{C}$. It turns out that this process can be described as a thermodynamic cycle with the ideal gas. Using the given information, compute the efficiency of a cyclone as a thermodynamic heat engine with water as the working substance.

Vítek watched news.

The task tells us the cycle consists of four processes: water evaporation, rising of the steam, phase transition in the clouds and precipitation. Processes 2 and 4, i. e. rising and descent, are both very fast. Therefore we will neglect any heat exchange and consider these two processes as adiabatic ($\Delta Q = 0$). We've been also told that the engine works between two constant temperatures. This implies that the other two processes, 1 and 3, are isothermal. A thermodynamic cycle consisting of two isothermal and two adiabatic processes is called the Carnot cycle. And for the Carnot cycle the efficiency is given by

$$\eta = 1 - \frac{T_2}{T_1} \doteq 0.12\,.$$

The efficiency is 0.12.

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Problem FoL.19 ... rain 3 points

It started raining and the Fykos bird noticed that the density of raindrops (hits per unit area) on the western side of the faculty building is twice as large as on the southern side. Determine the direction of wind – specifically, the angle between this direction and the northward direction, in degrees. Assume the sides of the building are exactly facing the four cardinal directions.

Matěj likes to sit inside college during the rain.

Let us denote the density of rain droplets on the southern side by σ. Density on the western side is then 2σ. We want to find the direction along which the densities become seemingly identical.

Assume that wind blowing in one of the cardinal directions results in density σ_0 on the respective side of the building. An inclination of α decreases this density to $\sigma_0 \cos\alpha$ since the projected area of the wall is multiplied by $\cos\alpha$ and therefore receives ($\cos\alpha$)-times as much raindrops.

If the angle α is measured from the northern direction, the density of droplets will be

$$\sigma_0 \cos\alpha$$

on the southern wall and

$$\sigma_0 \cos(90^\circ - \alpha) = \sigma_0 \sin\alpha$$

on the western wall. Thus we obtain a set of linear equations

$$\sigma = \sigma_0 \cos\alpha \,,$$
$$2\sigma = \sigma_0 \sin\alpha \,.$$

solved by

$$\alpha = \arctan 2 \doteq 63.43^\circ \,.$$

As one would expect, the wind blows from the east with a slight inclination to the north.

Matěj Mezera
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Problem FoL.20 ... throwing balls — 4 points

Mišo and Dano are standing next to each other. At the same time, each of them throws a tennis ball with the same velocity $v = 15\,\mathrm{m{\cdot}s^{-1}}$. Dano throws his ball under an elevation angle $\alpha_1 = 45^\circ$ and Mišo throws his under an elevation angle $\alpha_2 = 35^\circ$. How much farther will the winning ball go? Assume that the balls fly next to each other in parallel planes and start from zero height. Neglect air resistance. *Danka remembered physical education lessons.*

First we decompose the velocity of the tennis ball into perpendicular components

$$v_x = v_0 \cos\alpha \,,$$
$$v_y = v_0 \sin\alpha \,.$$

Choosing the coordinate system so that the initial postion of a ball lies in its origin (initial elevation is zero), we can express the time dependence of a ball's coordinates as

$$x = v_x t \,,$$
$$y = v_y t - \frac{1}{2} g t^2 \,.$$

Now we want to find the maximum distance d. For time of flight t_d we have

$$x_d = d = v_x t_d \,,$$
$$y_d = 0 = v_y t_d - \frac{1}{2} g t_d^2 \,.$$

We solve the first equation for t_d,

$$t_d = \frac{d}{v_x}$$

and substitute into the second equation. After few simple manipulations we obtain

$$d = \frac{2 v_x v_y}{g} \,,$$
$$d = \frac{2 v^2 \sin\alpha \cos\alpha}{g} \,,$$
$$d = \frac{v^2}{g} \sin 2\alpha \,.$$

To compute the difference between ranges d_1 and d_2 for each ball, we just subtract d_2 from d_1 because we know that without air drag the distance d_1 (45° elevation angle) must be larger.

$$\Delta d = d_1 - d_2 = \frac{v^2}{g} \left(\sin 2\alpha_1 - \sin 2\alpha_2\right) \doteq 1.38\,\mathrm{m}\,.$$

The winning ball will go 1.38 m farther than the other ball. However, in real conditions the ball thrown by Mišo might go further because air drag with v^2 dependence decreases the optimal elevation angle.

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Problem FoL.21 ... shadow

4 points

What is the smallest (absolute) latitude such that when you are standing on a horizontal surface at this latitude, your shadow (created by the Sun) will never be shorter than you?

Matěj is hiding in the shadows of the others.

Shadows cast by the sun are the shortest at high noon (Solar noon). First we solve the task for equinox and then we add the effect of inclination of the Earth's rotation axis. The angle between the vertical and the direction of the sun rays is φ, where φ denotes the geographic latitude. Using simple trigonometry we obtain the length of the shadow

$$s = h \tan\varphi\,,$$

where h is the height of the object (i. e. us). We are interested only in the critical case of $s = h$. Then we have

$$\tan\varphi = 1\,,$$
$$\varphi = 45^\circ\,.$$

Now we include inclination of the rotation axis $\Phi = 23.4^\circ$. In the worst case scenario, that is during solstice, the angle between sun rays and the vertical is going to be $\varphi - \Phi$. So

$$\varphi - \Phi = 45^\circ\,,$$
$$\varphi = 45^\circ + 23.4^\circ = 68.4^\circ\,.$$

This means that at latitudes higher than this value, our shadow can never be shorter than us. The result accidentally almost coincides with the latitude of the polar circle $\varphi \approx 66.5^\circ$.

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Problem FoL.22 ... elevator acceleration

4 points

An elevator should travel from the ground level of a building to the top at height 60 m*. During the acceleration phase, its acceleration must not exceed* $4\,\mathrm{m{\cdot}s^{-2}}$*; when it decelerates, it must not exceed* $6\,\mathrm{m{\cdot}s^{-2}}$ *(in absolute value). What is the minimum time the elevator needs to reach the top of the building?*

Are there any limits to the optimization of the elevator usage?

The fastest operation regime is achieved when the elevator accelerates as much as possible for time t_1 and then immediately starts decelerating. Let's denote the deceleration time as t_2. Initially the elevator is still and it again comes to a halt at the end of the travel. Therefore

$$v = a_1 t_1 = a_2 t_2 \,.$$

We can rewrite this equation to relate the two times,

$$t_2 = \frac{a_1}{a_2} t_1 \,.$$

The height (distance travelled under constant acceleration) is given by the well known formula

$$h = \frac{1}{2} a_1 t_1^2 + \frac{1}{2} a_2 t_2^2 = \frac{a_1 (a_1 + a_2)}{2 a_2} t_1^2 \,.$$

This leads to

$$t_1 = \sqrt{\frac{2 h a_2}{a_1 (a_1 + a_2)}} \,,$$

and similarly,

$$t_2 = \sqrt{\frac{2 h a_1}{a_2 (a_1 + a_2)}} \,.$$

The total time is then given by the sum

$$t = t_1 + t_2 = \sqrt{\frac{2h (a_1 + a_2)}{a_1 a_2}} \doteq 7.07\,\mathrm{s} \,.$$

The minimum time of travel is $7.07\,\mathrm{s}$.

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Problem FoL.23 ... Kevin the Cube

5 points

A large homogeneous cube with side length $a = 15\,\mathrm{m}$ decided to roll around one of its edges on a horizontal surface in such a way that the energy used for this would be the minimum possible. Unfortunately, at distance a from this egde, there is a tomato that is soon to be squashed. With what velocity will an edge of the cube hit the tomato? The acceleration due to gravity is $g = 9.81\,\mathrm{m{\cdot}s^{-2}}$. *Matěj is fascinated by rolling balls.*

During the rolling motion the center of mass of the cube will reach its maximum height of $\frac{\sqrt{2}}{2}a$. After reaching the maximum, the following ("falling") motion will decrease the potential energy of the cube by

$$\Delta E = \frac{1}{2} \left(\sqrt{2} - 1 \right) amg \,,$$

where m is the mass of Kevin the Cube. According to the Steiner theorem, the moment of inertia of a cube rolling around its edge is

$$J = \frac{1}{6} ma^2 + \frac{1}{2} ma^2 = \frac{2}{3} ma^2 \,.$$

Now from the kinetic energy formula $E_\mathrm{k} = \frac{1}{2}J\omega^2$ we can easily obtain

$$v = a\omega = a\sqrt{\frac{2\Delta E}{J}} = a\sqrt{\frac{\left(\sqrt{2}-1\right)amg}{\frac{2}{3}ma^2}} = \sqrt{\frac{3(\sqrt{2}-1)}{2}ag} \doteq 9.562\,\mathrm{m{\cdot}s^{-1}}\,.$$

The edge will smash the tomato with velocity $9.56\,\mathrm{m{\cdot}s^{-1}}$.

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Problem FoL.24 ... chained

5 points

A chain with mass 6.0 kg *and length* 11 m *is hanging down from one end at height* 15 m *above the ground. Find the maximum kinetic energy of the chain after it is released. Consider that the chain hits the ground perfectly inelastically.*

Josef Jírů wanted to enrich our participants with a problem with a chain.

Let's denote the mass of the chain as m, its length as l, the height from the upper end above the ground as h and x is the distance of the chain link, that has just hit the ground, from the chain's upper end. If the entire chain is moving, its kinetic energy E_k is increasing. At the instant of impact, the kinetic energy equals $E_\mathrm{k} = mg(h-l) = 235\,\mathrm{J}$. Later, the kinetic energy of the chain for $x \in \langle 0, l\rangle$ is given by

$$E_\mathrm{k}(x) = \frac{1}{2}\frac{x}{l}m2g(h-x) = \frac{mg}{l}\left(-x^2 + hx\right)\,.$$

The expression in the brackets is a quadratic function with zeroes 0 and h, the function attains its maximum for $x = h/2$. For the maximum energy, one obtains

$$E_\mathrm{kmax} = E_\mathrm{k}(h) = \frac{mgh^2}{4l} \doteq 301\,\mathrm{J}\,,$$

which is, indeed, a higher value than 235 J.

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Problem FoL.25 ... cork problem

4 points

Vítek was investigating some fluid properties using a water surface with surface tension $\sigma_1 = 70\,\mathrm{mN{\cdot}m^{-1}}$*. For this purpose, he made a thin cork board with mass* $m = 100\,\mathrm{g}$ *and side lengths* $b = 5\,\mathrm{cm}$ *and* $c = 7\,\mathrm{cm}$*, which was – to his surprise – floating on the surface. He decided to get the board moving, so he added a bit of detergent to one of its shorter sides. The surface tension of the resulting soap (detergent) solution is* $\sigma_2 = 40\,\mathrm{mN{\cdot}m^{-1}}$*. Help Vítek find the acceleration of the cork board. For simplicity, assume that water resistance is negligible and that the contact angles are always* 90°*.*

Vítek likes to accelerate cork with some detergent.

We dump the detergent to the shorter edge of the board with a length of b. The formed soapsuds have different surface tension σ_2 and therefore the board is no longer in the equilibrium. The forces acting on sides with a length of c will cancel out. The net force is given by

$$F_1 - F_2 = b(\sigma_1 - \sigma_2) = ma \qquad \Rightarrow \qquad a = \frac{b(\sigma_1 - \sigma_2)}{m}\,.$$

If we evaluate this equation, we obtain $a = 1.5 \cdot 10^{-2}\,\mathrm{m{\cdot}s^{-2}}$.

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Problem FoL.26 ... olive one 6 points

An ideal, seedless olive may be modelled by a hollow sphere with inner and outer radii $r_1 = 3.00\,\mathrm{mm}$ and $r_2 = 10.0\,\mathrm{mm}$, respectively. If the density of an ideal olive is $\varrho_\mathrm{o} = 816\,\mathrm{kg{\cdot}m^{-3}}$, how deep will it submerge in brine with density $\varrho_\mathrm{b} = 1{,}211\,\mathrm{kg{\cdot}m^{-3}}$? Assume that an ideal olive is perfectly tasty and that brine fills the hole created by the missing seed.

Jáchym is attracted by olives.

Let $V(x)$ denote the volume from the bottom of the olive up to a distance x. The volume of the whole olive is given by

$$V(2r_2) = \frac{4}{3}\pi\left(r_2^3 - r_1^3\right) .$$

Archimedes' principle tells us that the volume $V(h)$ submerged in the brine may be written as

$$V(h) = \frac{\varrho_\mathrm{o}}{\varrho_\mathrm{b}} V(2r_2) .$$

Let's divide $V(x)$ into three separate functions on three disjoint intervals. For $x \in \langle 0, r_2 - r_1\rangle$, we may write

$$V(x) = \int_0^x S(y)\mathrm{d}y ,$$

where the cross-sectional area at height y above the bottom of the olive is given by

$$S(y) = \pi r^2(y) = \pi\left(r_2^2 - (r_2 - y)^2\right) = \pi\left(2r_2 y - y^2\right) .$$

Substituting for the integrand, we get the well-known formula for the volume of a spherical cap

$$V(x) = \pi\int_0^x \left(2r_2 y - y^2\right)\mathrm{d}y = \pi\left[r_2 y^2 - \frac{y^3}{3}\right]_0^x = \pi\left(r_2 x^2 - \frac{x^3}{3}\right) .$$

Obviously, $V(x) < V(h)$ for all $x \in \langle 0, r_2 - r_1\rangle$, since $V(h) > V(2r_2)/2 = V(r_2)$. Using symmetry, we could calculate $V(x)$ for $x \in \langle r_2 + r_1, 2r_2\rangle$, but it can be shown that these volumes would be too large. Therefore, we need to find a formula for $V(x)$ with x in the middle interval $(r_2 - r_1, r_2 + r_1)$. Again, this is not particularly difficult, we can just subtract the volume of a smaller spherical cap from the volume $V(x)$ obtained in the same way as for the first interval, that is

$$\begin{aligned} V(x) &= \pi\left(r_2 x^2 - \frac{x^3}{3}\right) - \pi\left(r_1\left(x - r_2 + r_1\right)^2 - \frac{\left(x - r_2 + r_1\right)^3}{3}\right) \\ &= \pi x\left(r_2^2 - r_1^2\right) - \pi\left(r_2 - r_1\right)^2 \frac{r_2 + 2r_1}{3} . \end{aligned}$$

Finally, we calculate the height

$$h = \frac{3V(h) + \pi\left(r_2 + 2r_1\right)\left(r_2 - r_1\right)^2}{3\pi\left(r_2^2 - r_1^2\right)} = \frac{4\frac{\varrho_\mathrm{o}}{\varrho_\mathrm{b}}\left(r_2^3 - r_1^3\right) + \left(r_2 + 2r_1\right)\left(r_2 - r_1\right)^2}{3\left(r_2^2 - r_1^2\right)} \doteq 12.5\,\mathrm{mm} .$$

A perfect olive will submerge $h \doteq 12.5\,\mathrm{mm}$ deep in the brine.

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Problem FoL.27 ... Paťo is doing sports 5 points

Legend says Paťo finished a twelve-minute run in six minutes. Along the track, there are milestones at equal distances $\Delta l = 5.000\,000\,000\,\mathrm{km}$; the first milestone is Δl from the starting point. Find the number of milestones Paťo passed if his watch showed that he ran for six minutes when he finished the run. The speed of light is $299{,}792{,}458\,\mathrm{m{\cdot}s^{-1}}$.

Every evening, Lukáš looks under his bed to make sure Paťo isn't there.

The difference in time readings performed by Paťo and by a referee is due to time dilation. If we denote Paťo's proper time by $\Delta\tau = 6\,\mathrm{min}$ and the referee's time by $\Delta t = 12\,\mathrm{min}$, the relation for the time dilation reads

$$\Delta t = \frac{\Delta\tau}{\sqrt{1-\left(\frac{v}{c}\right)^2}}\,.$$

From this formula we work out the velocity

$$v = c\sqrt{1-\left(\frac{\Delta\tau}{\Delta t}\right)^2}\,,$$

so Paťo traveled distance (observed from the reference frame of milestones)

$$l = v\Delta t = c\Delta t\sqrt{1-\left(\frac{\Delta\tau}{\Delta t}\right)^2} \doteq 186{,}932{,}077\,\mathrm{km}\,.$$

This amounts to 37,386,415 passed milestones (the ratio $\Delta l/l$ was rounded down to an integer value).

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Problem FoL.28 ... energetic pendulum 5 points

A mathematical pendulum is a small ball with mass $m = 5\,\mathrm{g}$ hanging from a (massless) rope with length $l = 2\,\mathrm{m}$. We displace the ball from its equilibrium position by an angle $\alpha = 4^\circ 10'$ and let it oscillate. After completing one swing (half-oscillation), the ball retains $99\,\%$ of the mechanical energy it had before that swing. After how many full swings (half-oscillations) will the velocity of the ball when passing through the equilibrium position decrease under $v_h = 0.2\,\mathrm{m{\cdot}s^{-1}}$? Assume the pendulum loses its energy symmetrically with respect to the equilibrium position.

Daniela closely watched the "renovation" of student cafeteria in Troja.

Let's set ground level of potential energy to the level of ball's equilibrium position. After displacing the pendulum by an angle α, the ball obtains the initial potential energy

$$E_{p_0} = mgh_0\,,$$

while h_0 satisfies

$$h_0 = l\left(1 - \cos\alpha\right) \,.$$

During the oscillations, potential energy changes to kinetic and vice versa, while part of the energy dissipates. Let's denote $\eta = 0.99$. After one swing, the potential energy of the ball equals

$$E_{p_1} = \eta E_{p_0} \,.$$

Let's denote n number of swings, after which the condition from assignment is satisfied. Then after n swings, the potential energy in an amplitude equals

$$E_{p_n} = \eta^n E_{p_0} \,.$$

When the ball passes the equilibrium position for the first time, it has zero potential energy and it's kinetic energy equals

$$E_{k_1} = \frac{1}{2} m v_1^2 \,,$$

while the equation

$$E_{k_1} > E_{p_1}$$

is satisfied. During n-th passing through the equilibrium position it therefore holds

$$\begin{aligned} E_{k_n} &> E_{p_n} \,, \\ \frac{1}{2} m v_h^2 &> \eta^n m g h_0 \,, \\ n &> \log_\eta \frac{v_h^2}{2gl\left(1 - \cos\alpha\right)} \,, \\ n &> 94.8 \,. \end{aligned}$$

So the pendulum must perform 95 full swings.

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Problem FoL.29 ... Einstein with Cherenkov 6 points

A photon collides with a gold surface submerged in a medium with refractive index $n = 1.804{,}17$ and ejects an electron. This electron is so fast that in this medium, it is moving faster than the speed of light in this medium. What is the maximum possible wavelength of the photon? The work function of gold is $W_0 = 5.45000\,\mathrm{eV}$. The speed of light in vacuum is $c = 2.997{,}925 \cdot 10^8\,\mathrm{m{\cdot}s^{-1}}$, Planck constant $h = 6.626{,}07 \cdot 10^{-34}\,\mathrm{J{\cdot}s}$, electron rest mass $m_0 = 9.109{,}38 \cdot 10^{-31}\,\mathrm{kg}$ and elementary charge $e = 1.60218 \cdot 10^{-19}\,\mathrm{C}$.

Karel was looking into a nuclear reactor.

A photon with frequency f and wavelength λ traveling in medium with refractive index n has energy

$$E = hf = \frac{hc}{n\lambda} \,. \tag{1}$$

The resulting energy of the ejected electron is calculated as

$$W = E - W_0 \,.$$

Then the velocity of the electron must be at least

$$v = \frac{c}{n}$$

and this velocity is used for computation of relativistic kinetic energy

$$W = m_0 c^2 \left(\frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} - 1 \right) = m_0 c^2 \left(\frac{n}{\sqrt{n^2 - 1}} - 1 \right) .$$

Now we use (1) and easily obtain the maximum wavelength of the electron

$$\lambda = \frac{hc}{nE} = \frac{hc}{n\left(W_0 + W\right)} = \frac{hc}{n\left(W_0 + m_0 c^2 \left(\frac{n}{\sqrt{n^2-1}} - 1\right)\right)} \doteq 6.675{,}82 \cdot 10^{-12}\,\mathrm{m} = 6.675{,}82\,\mathrm{pm}\,.$$

The maximum wavelength is 6.675,82 pm.

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Problem FoL.30 ... two-state problem — 5 points

The Fykos bird started learning statistical physics, but he cannot solve the following introductory problem. Consider N particles in a closed box. Let's split them into two groups – particles in the right half of the box are in state 0 and those in the left half are in state 1. We do not care about which individual particles have which state, only that N_0 have state 0 and N_1 have state 1 ($N_1 + N_0 = N$). Microstates of the system are described by a number $m = N_1 - N_0$, which describes the distribution of particles between the two states. Let's denote the number of ways to split the particles into the two states by $t(m)$ (e.g. for $N = m = 10$, $t = 1$). Note that the exact analytical formula may be simplified to

$$t(m) = \sqrt{\frac{2}{\pi N}} 2^N \exp\left(-\frac{m^2}{2N}\right)$$

using Stirling approximation. Evaluate both formulas (exact and approximate) for $N = 20$ and $m = 8$ and find the relative error (in percentage) of the approximate value with respect to the exact value, i.e. by how many percent the approximate value differs from the exact one.

Vítek is in another state.

To solve this task we must realize that the distribution described above is binomial – the problem is the same as e. g. "find the probability of getting 8 heads in 20 coin tosses". Our job is to calculate the binomial coefficient for the event N_1 within N possible states (or even N_1, since the probabilities for 0 and 1 are the same, that is 1/2). From the definition of binomial coefficient,

$$t = \binom{N}{N_1} = \frac{N!}{(N - N_1)!N_1!} ,$$

which is the number of ways of choosing N_1 particles from the set of N particles. The expression $N - N_1$ is exactly N_0. Substituting $N_0 = (N - m)/2$ and $N_1 = (N + m)/2$ we get

$$t = \frac{N!}{\left(\frac{N-m}{2}\right)!\left(\frac{N+m}{2}\right)!}\,.$$

Now we plug in the number into this formula and also into the approximate one. The resulting difference is

$$q = 1 - \frac{37{,}771}{38{,}760}\,,$$

which can be also written as 2.55 %.

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Problem FoL.31 ... spacey — 5 points

In a lab on Earth, we measure the radiation spectrum of a star that is moving away from Earth with velocity $v = 1.70 \cdot 10^7\,\mathrm{m{\cdot}s^{-1}}$. The maximum intensity in this spectrum is at wavelength $\lambda_0 = 700\,\mathrm{nm}$. Determine the surface temperature of the star. Assume that the star is an ideal black body. The speed of light is $c = 3.00 \cdot 10^8\,\mathrm{m{\cdot}s^{-1}}$.

Vítek was reading Fundamental Astronomy.

Given that the star is receding the Doppler redshift must apply. The relation between emitted and observed wavelength reads $\lambda_0 = \lambda_\mathrm{e}(1 + z)$, where z is the redshift factor. According to Wien's law a black body with a temperature T has radiation peak at wavelength λ_m (in the reference frame of the star) given by

$$\lambda_\mathrm{m} = \frac{b}{T}\,,$$

where $b = 2.898\,\mathrm{mm{\cdot}K}$ is Wien's constant and λ_m is identified as λ_e.

Redshift can be expressed in the terms of receding velocity as

$$z = \left(\frac{1 + \frac{v}{c}}{1 - \frac{v}{c}}\right)^{\frac{1}{2}} - 1\,,$$

where c is the speed of light. Combining all formulas we determine the temperature as

$$T = \frac{b}{\lambda_0}\left(\frac{1 + \frac{v}{c}}{1 - \frac{v}{c}}\right)^{\frac{1}{2}}\,.$$

For given numerical values we arrive at $T \doteq 4{,}380\,\mathrm{K}$.

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Problem FoL.32 ... one-third train 3 points

A train was moving with velocity $96\,\mathrm{km\cdot h^{-1}}$. *It engaged its brakes for the first time at time* $99\,\mathrm{s}$ *before coming to a halt. What was the distance at which it engaged the brakes? During the first* $1/3$ *of this distance, the train was decelerating with constant acceleration and decreased its velocity to* $1/2$. *During the second* $1/3$ *of the same distance, it was moving with constant velocity, and during the last* $1/3$, *it was again decelerating with constant acceleration until it stopped.* *Josef Jírů wanted to enrich our participants with a problem with trains.*

Let v_0 denote the speed of the train before engaging the brakes and s_1 is $1/3$ of the distance mentioned above. Different times, that the train spent in each $1/3$ of the distance, are in a ratio of

$$t_1 : t_2 : t_3 = \frac{s_1}{\frac{3}{4}v_0} : \frac{s_1}{\frac{1}{2}v_0} : \frac{s_1}{\frac{1}{4}v_0}\,,$$

where denominators of right hand side of equation above equal average speed of the train in given $1/3$ of the distance (we assume $v_{\mathrm{avg}} = s_1/t_{1,2,3}$). Also, right hand side may be written as

$$\frac{s_1}{\frac{3}{4}v_0} : \frac{s_1}{\frac{1}{2}v_0} : \frac{s_1}{\frac{1}{4}v_0} = \frac{4}{3} : 2 : 4 = 2 : 3 : 6\,.$$

E.g. time, that the train spent in second third of the distance, equals $t_2 = 3/11 \cdot 99\,\mathrm{s} = 27\,\mathrm{s}$, which means that it covered distance $s_1 = v_0 t_2/2 = 360\,\mathrm{m}$. Therefore the total distance is three times s_1, which means $1{,}080\,\mathrm{m}$.

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Problem FoL.33 ... marble on a spiral 5 points

There is a spiral defined by parametric equations $x = \sin\varphi$, $y = -\cos\varphi$, $z = \varphi/2\pi$, *placed in a homogeneous gravitational field* $g = 9.81\,\mathrm{m\cdot s^{-2}}$ *in the direction of the negative* z*-axis. We place a marble on the spiral. The marble does not rotate and moves on the spiral without friction. What will the negative* z*-component of its acceleration be?*

Karel likes to play with marbles.

In the xy plane, the spiral looks like a unit circle. We are interested only in z compound of the acceleration. Because the force, causing the marble to follow a curved path, is perpendicular to the direction of motion and therefore does no mechanical work, we can unwrap the whole spiral into a straight line. By doing this, we transform the issue into the problem of an inclined plane. Let α denote the angle between the inclined and horizontal plane. With every circle travelled the marble changes its height by 1. Therefore we obtain

$$\tan\alpha = \frac{1}{2\pi}\,,$$

$$\alpha = \arctan\frac{1}{2\pi} \doteq 0.16\,.$$

The compound of gravitational acceleration, affecting the marble in the direction of its movement, will be $g\sin\alpha$. However, we are interested in the marble's acceleration in the direction of z axis. To calculate this, we multiply the previous result once more by $\sin\alpha$. We obtain

$$a = g\sin^2\alpha = g\sin^2\arctan\frac{1}{2\pi} = \frac{g}{1+(2\pi)^2}\,,$$

what is approximately $0.242\,\mathrm{m\cdot s^{-2}}$.

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Problem FoL.34 ... olive two 6 points

An ideal olive is a perfect sphere containing a spherical seed with radius $r_1 = 3\,\mathrm{mm}$. The density of an ideal olive is equal to the density of brine $\varrho_\mathrm{n} = 1{,}211\,\mathrm{kg\cdot m^{-3}}$ so that it would freely float in a can. Consider an open cylindrical can of olives in brine with base radius $R = 31\,\mathrm{mm}$. If we remove the seed from an olive, the rest of the olive floats up to the surface of the brine and this surface decreases by $\Delta h = 4\cdot 10^{-5}\,\mathrm{m}$ in the process. What is the density of the seed? Assume that the brine fills the hole created by removing the seed, too.

Jáchym cannot help himself and still thinks just about olives.

Let r_2 denote radius of the olive. Total density of the perfect olive is ϱ_n. If we mark density of the seed as ϱ_p and of the rest of the olive ϱ_o, we obtain equation

$$\frac{4}{3}\pi\left(r_2^3 - r_1^3\right)\varrho_\mathrm{o} + \frac{4}{3}\pi r_1^3 \varrho_\mathrm{p} = \frac{4}{3}\pi r_2^3 \varrho_\mathrm{n}\,,$$

from which we can express

$$\varrho_\mathrm{p} = \frac{r_2^3\varrho_\mathrm{n} - \left(r_2^3 - r_1^3\right)\varrho_\mathrm{o}}{r_1^3}\,.$$

When the rest of the olive will be in rest on the surface, we will mark the volume above and under surface as V_1 and V_2, respectively. Then

$$V_1 + V_2 = \frac{4}{3}\pi\left(r_2^3 - r_1^3\right)\,,$$
$$V_2\varrho_\mathrm{n} = \left(V_1 + V_2\right)\varrho_\mathrm{o}\,,$$

from where we can express

$$V_1 = \frac{4}{3}\pi\left(r_2^3 - r_1^3\right)\left(1 - \frac{\varrho_\mathrm{o}}{\varrho_\mathrm{n}}\right)\,.$$

Total volume under that surface decreased by volume of the seed and also by volume V_1. This volume is nercessarily equal to the volume, that decreased in the can. That one we can express as $\pi R^2\Delta h$. From this equality we find out that

$$\pi R^2\Delta h = V_1 + \frac{4}{3}\pi r_1^3 = \frac{4}{3}\pi\left(r_2^3 - r_1^3\right)\left(1 - \frac{\varrho_\mathrm{o}}{\varrho_\mathrm{n}}\right) + \frac{4}{3}\pi r_1^3\,,$$

from which we express

$$\varrho_\mathrm{o} = \left(1 - \frac{3R^2\Delta h - 4r_1^3}{4\left(r_2^3 - r_1^3\right)}\right)\varrho_\mathrm{n}\,.$$

Adding into one of equations found above we obtain the result

$$\varrho_\mathrm{p} = \frac{3R^2\Delta h}{4r_1^3}\varrho_\mathrm{n} \doteq 1{,}293\,\mathrm{kg\cdot m^{-3}}\,.$$

Density of a seed of a perfect olive is $\varrho_\mathrm{p} \doteq 1{,}293\,\mathrm{kg\cdot m^{-3}}$.

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Problem FoL.35 ... Dan's eggs
7 points

Dan wants to throw a raw egg to Danka in such a way that she can catch it at minimal velocity (in order to minimise the probability that it breaks when caught). He's throwing from a balcony at height h, Danka is at height 0. Due to bad terrain, Danka can only stand at a distance greater than $3h$ from the base of the balcony. Find the elevation angle under which Dan should throw the egg. Do not consider air resistance. *Girls like to grab eggs.*

We choose a coordinate system in which y axis is vertical and points up to Dan, with zero being at the ground level. A throw at Danka's position conforms to equations

$$
\begin{aligned}
3h &= v_0 t \cos\alpha \,, \\
0 &= h + v_0 t \sin\alpha - \frac{1}{2} g t^2 \,.
\end{aligned}
$$

Factoring out the time variable we get

$$0 = h + 3h\tan\alpha - \frac{9gh^2}{2v_0^2\cos^2\alpha} \,.$$

It follows

$$v_0^2 = \frac{9gh}{2\cos^2\alpha\,(1 + 3\tan\alpha)} = \frac{9gh}{2\cos^2\alpha + 3\sin 2\alpha} \,.$$

Now we apply energy conservation and seek for an angle that minimizes v_0. Then the impact velocity is minimized as well. In other words, the denominator must be maximized by this angle. Derivative of the denominator with respect to α is

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\left(2\cos^2\alpha + 3\sin 2\alpha\right) = -2\sin 2\alpha + 6\cos 2\alpha \,.$$

Equating this to zero results in $\tan 2\alpha = 3$. In the interval $(-\pi/2, \pi/2)$ there are two roots, $\alpha_1 \doteq 35.8^\circ$ and $\alpha_2 \doteq -54.2^\circ$. However, for the second value the initial velocity is imaginary, so the correct physical result is 35.8°.

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Problem FoL.36 ... a motivational capacitor
6 points

Vítek has a favourite parallel-plate capacitor. Its plates are separated by distance $d = 2.0\,\mathrm{mm}$ and the area of each plate is $S = 20.0\,\mathrm{cm}^2$. He placed a charge of Q on one plate and a charge of $-Q$ on the other plate, where $Q = 0.1\,\mathrm{nC}$. Then he decided to fill the space between the plates with a dielectric material. As a result, the electric intensity between the plates is no longer constant, but given by the formula

$$E(x) = -\frac{Q\mathrm{e}^{x/d}}{\varepsilon_0 \kappa_0 S} \,,$$

where $\kappa_0 = 1.5$ and x is the distance from the negatively charged plate. Find the capacitance of this capacitor. *Vítek broke a capacitor at his practical class.*

We are dealing with a 1D problem, the electric field can be thus written in the form $E(x) = -\frac{\partial\varphi}{\partial x}$. Capacitance of the capacitor C is defined as $C = Q/\Delta\varphi$, where $\Delta\varphi$ gives the potential difference between the plates and Q is the magnitude of the charge that each plate was given. $E(x)$ is given in the task, so we need to integrate a differential equation

$$\mathrm{d}\varphi = \frac{Q}{\varepsilon_0 \varkappa_0 S} \mathrm{e}^{x/d} \,\mathrm{d}x \,.$$

$$\int_{\varphi_1}^{\varphi_2} \mathrm{d}\varphi = \frac{Q}{\varepsilon_0 \varkappa_0 S} \int_0^d \mathrm{e}^{\frac{x}{d}} \,\mathrm{d}x \,,$$

$$\Delta\varphi = \frac{Qd}{\varepsilon_0 \varkappa_0 S} \left(\mathrm{e} - 1\right) \,.$$

Substituting the result into the definition of capacitance, we get

$$C = \frac{\varepsilon_0 \varkappa_0 S}{d\left(\mathrm{e} - 1\right)} \doteq 7.7\,\mathrm{pF} \,.$$

As we can see, the result is independent of the stored charge. This was to be expected because capacitance is a characteristic of electronic components.

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Problem FoL.37 ... a calm place 5 points

In a vast, empty country, there are two very long, straight, parallel roads $s = 490.0\,\mathrm{m}$ apart. Matěj found himself between these two roads. However, he cannot cross the first road, because there are $n_1 = 0.90$ cars per second flowing through it. The second road has even more traffic, cars flow through it at a rate of $n_2 = 1.60$ cars per second. The velocity of cars is the same on both roads. He had no choice but to wait for death, since there was no crossing in sight. Therefore, he found a place where the total noise from both roads was at a minimum, sat down, and waited... How far from the first road is he sitting? There are no sound barriers, sound travels through the air isotropically and without losses and each car is equally noisy.

Matěj got this idea when he wanted to kill himself after not acing an exam.

How does the noise coming from one road depend on perpendicular distance? A point sound source emits noise isotropically, its intensity I in distance r is $I(r) = K/r^2$, where K is a constant dependent on the intensity of the source.[1] However, the road is not a point source but a line source. If we recall Gauss's law, we can deduce the formula for the intensity of this source $I(r) = k'/r$, where k' is another, unimportant constant. Otherwise, we can integrate the road as a continuous set of point sources

$$I = \int\limits_{-\infty}^{\infty} \frac{k}{y^2 + r^2} \,\mathrm{d}y = \int\limits_{-\infty}^{\infty} \frac{k}{r} \frac{1}{\left(\frac{y}{r}\right)^2 + 1} \frac{\mathrm{d}y}{r} = \frac{k}{r} \int\limits_{-\infty}^{\infty} \frac{1}{t^2 + 1} \,\mathrm{d}t = \frac{k}{r} \left[\arctan t\right]_{-\infty}^{\infty} = \frac{\pi k}{r} \,,$$

where we substituted $y/r = t$ and k is yet another constant, which depends on the linear intensity density of the source.

[1] In the solution we deal with numerous constants whose value is not important to us. We denote those constant as various types of the letter K.

An important property of the sound intensity is its additivity. That is, the total intensity is given by the sum of intensities from all sources (this was already used in the integration). One can easily observe that k and k' are directly proportional to the car flow of respective roads, $k' = \pi k = \mathcal{K}n$.

In the following, the unknown distance of Matěj from the first road is denoted by x. The intensity in distance x is obtained as a sum of intensities from each road,

$$I_{\mathrm{tot}}(x) = \frac{\mathcal{K}n_1}{x} + \frac{\mathcal{K}n_2}{s-x}\,.$$

Now, we need to minimize this function, which is done with the use of its first derivative with respect to x.

$$\begin{aligned}\frac{\mathrm{d}I_{\mathrm{clk}}(x)}{\mathrm{d}x} &= 0\\ -\frac{\mathcal{K}n_1}{x^2} + \frac{\mathcal{K}n_2}{(s-x)^2} &= 0\\ n_1(s-x)^2 &= n_2x^2\\ \left(\frac{s}{x}-1\right)^2 &= \frac{n_2}{n_1}\\ \frac{s}{x} &= 1+\sqrt{\frac{n_2}{n_1}}\\ x &= s\frac{n_1\sqrt{n_1n_2}}{n_1-n_2}\end{aligned}$$

where the obvious inequality $s/x > 1$ is utilized. Substitution of known values gives us

$$x = \frac{3}{7}s = 210\,\mathrm{m}\,.$$

The second solution, with minus sign, has no physical meaning because $I_{\mathrm{tot}}(x)$ is defined only on the interval between the two roads. This way we avoided the absolute values $|x|$ and $|s-x|$.

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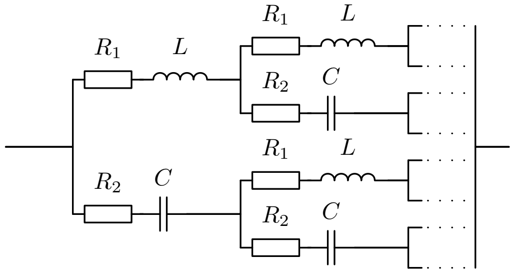
Problem FoL.38 ... finally a complex infinite circuit 5 points

Consider the infinite RLC circuit in figure. The inductance of each coil is $L = 3\,\mathrm{mH}$, the capacity of each capacitor is $C = 4{,}700\,\mathrm{nF}$ and the resistances are $R_1 = 1.5\,\mathrm{k\Omega}$ and $R_2 = 0.3\,\mathrm{k\Omega}$. At which frequency of alternating current is the impedance of the circuit purely real? Neglect phase shift caused by finite propagation speed of electromagnetic field.

Jáchym likes to be complex.

If we denote the impedances of respective components as

$$\begin{aligned}Z_L &= \mathrm{i}\omega L\,,\\ Z_C &= \frac{1}{\mathrm{i}\omega C}\,,\\ Z_{R_1} &= R_1\,,\\ Z_{R_2} &= R_2\,,\end{aligned}$$



then for the impedance of the whole circuit we have

$$Z = \frac{(Z_L + Z_{R_1} + Z)(Z_C + Z_{R_2} + Z)}{Z_L + Z_{R_1} + Z + Z_C + Z_{R_2} + Z}\,.$$

Squaring and substitution results in

$$Z^2 = Z_L Z_C + Z_L Z_{R_2} + Z_C Z_{R_1} + Z_{R_1} Z_{R_2} = \frac{L}{C} + R_1 R_2 + \mathrm{i}\left(\omega L R_2 - \frac{R_1}{\omega C}\right)\,.$$

To have a real Z, we need real Z^2 as well, so

$$\omega L R_2 - \frac{R_1}{\omega C} = 0 \qquad \Rightarrow \qquad \omega = \sqrt{\frac{R_1}{R_2 L C}}\,.$$

The frequency is now easily computed from the definition as

$$f = \frac{\omega}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{R_1}{R_2 L C}} \doteq 3.00\,\mathrm{kHz}\,.$$

The impedance of the circuit is real for the frequency $f \doteq 3.00\,\mathrm{kHz}$.

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Problem FoL.39 ... vertical shelf 4 points

There is a copper cuboid with length of $l_0 = 1{,}970.0\,\mathrm{mm}$ and mass of $m = 79.00\,\mathrm{kg}$. If the temperature of copper decreases by $\Delta T = 12\,\mathrm{K}$, the cuboid perfectly fits between two rigid blocks (which do not expand or contract with changes in temperature). We wait for the copper to heat up to its original temperature. What mass may we place on the cuboid before it falls down? The static friction coefficient between copper and the blocks is $f = 0.15$. Assume that the cuboid does not deform when the mass is placed on it.

Jáchym thought up a new way to store things.

We denote the coefficient of linear thermal expansion of copper by α and express the distance between the two blocks as

$$l = l_0\left(1 - \alpha \Delta T\right)\,.$$

With Young's modulus of copper denoted by E, the force exerted by the cuboid on each side of the block is

$$F = SE\frac{l_0 - l}{l_0} = SE\alpha\Delta T\,,$$

where S denotes the cross-section of the cuboid, this value is determined from the known density ϱ of copper

$$S = \frac{m}{\varrho l_0}\,.$$

The maximum magnitude of the static friction force is $F_\mathrm{t} = 2fF$. The maximum mass M that can be placed on the cuboid is then

$$M = \frac{F_\mathrm{t}}{g} - m = m\left(\frac{2fE\alpha\Delta T}{\varrho l_0 g} - 1\right) \doteq 3{,}300\,\mathrm{kg}\,,$$

where we used the values $\alpha = 1.7\cdot 10^{-5}\,\mathrm{K^{-1}}$, $E = 120\,\mathrm{GPa}$ and $\varrho = 8{,}900\,\mathrm{kg{\cdot}m^{-3}}$. The cuboid can support an additional mass of 3,300 kg.

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Problem FoL.40 ... olive three

5 points

Jáchym is eating olives with eating speed proportional to the square of the number of uneaten olives. A day after he opened a can, there were 32 of them. A day later, there were just 17 left. How many olives were in the can initially? Assume that the number of olives is a continuous variable for Jáchym. *Jáchym dreams only about olives.*

Let's denote the number of olives by N. The eating rate is then

$$\frac{\mathrm{d}N}{\mathrm{d}t} = \lambda N^2\,.$$

A simple integration leads to

$$-\frac{1}{N} = \lambda t + C\,.$$

The task tells us that at time $t_1 = 1\,\mathrm{d}$, $N(t_1) = 32$ olives remained, while at time $t_2 = 2\,\mathrm{d}$ only $N(t_2) = 17$ olives remained. Thus, we have two equations for two unknowns. We solve the first for λ,

$$\lambda = -\frac{\frac{1}{N(t_1)} + C}{t_1}\,,$$

and substitute into the second equation:

$$-\frac{1}{N(t_2)} = -\frac{\frac{1}{N(t_1)} + C}{t_1}t_2 + C\,,$$

$$C = \frac{\frac{t_1}{N(t_2)} - \frac{t_2}{N(t_1)}}{t_2 - t_1}\,.$$

Now we can compute the result

$$N(0) = -\frac{1}{C} = \frac{t_1 + t_2}{\frac{t_2}{N(t_1)} - \frac{t_1}{N(t_2)}} = 272\,.$$

There are 272 olives left in the can.

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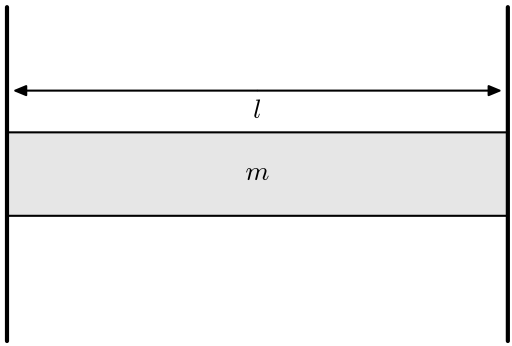
Problem FoL.41 ... elastic loop 6 points

A circular loop with radius $R_0 = 0.45\,\mathrm{m}$ is placed in a homogeneous static magnetic field with magnitude $B = 0.83\,\mathrm{T}$ in such a way that its axis is parallel with the direction of the magnetic field and the loop is compressed by the manetic field. There is an electric current $I = 16\,\mathrm{A}$ flowing through the loop. The loop is made from a brass wire with circular cross-section (with radius $r = 0.254\,\mathrm{mm}$) and Young's modulus $E = 98\,\mathrm{GPa}$. How does the length of the loop reduce (in mm)? *Vašek wanted to know what happens to a magnet in a magnetic field.*



The length of the loop changes from l_0 to l, that is $\Delta l = l - l_0$. Since the cross-section area of a wire is $S = \pi r^2$, the tension in the wire is given by the Hooke's law as

$$\sigma = \frac{\Delta l}{l_0} E = -\frac{F}{S}\,,$$

where F is the force acting inward on each element of the wire. Now, let's choose an infinitesimal section of the wire of length $\mathrm{d}x$. The central angle corresponding to this infinitesimal arc is

$$2\mathrm{d}\varphi = \frac{\mathrm{d}x}{R}\,,$$

where R is the radius of the loop, expressed using $l = 2\pi R$. The initial length of the loop was $l_0 = 2\pi R_0$.

Magnetic force $\mathrm{d}F_\mathrm{m}$ acts upon this element, pointing to the centre of the loop. Then, there is the force F acting on the element from each side. The element is at rest, therefore the forces must cancel out. Also, F can be decomposed into tangential and radial components. The tangential components cancel out each other and the radial component has opposite direction to the magnetic force. This radial force is related to the central angle by

$$\mathrm{d}F_x = F \sin \mathrm{d}\varphi \approx F \mathrm{d}\varphi\,.$$

It is also obvious that $2\mathrm{d}F_x = \mathrm{d}F_\mathrm{m}$ must hold.

The next to last step is to express the magnetic force in known variables. A particle with charge $\mathrm{d}q$ moving with velocity v in the magnetic field B experiences perpendicular force $\mathrm{d}F_m = \mathrm{d}qvB$. Velocity is defined as distance covered in time and electric current is defined as a flow of charge. This leads to

$$\mathrm{d}F_\mathrm{m} = \mathrm{d}qvB = \frac{\mathrm{d}q\,\mathrm{d}x}{\mathrm{d}t} = IB\,\mathrm{d}x\,.$$

Now we have all the ingredients we need to solve this problem. After substituting force F, expressed as

$$F = BIR\,,$$

into the formula for tension, we get

$$\sigma = \frac{\Delta l}{l_0}E = -\frac{BIR}{S}\,.$$

Finally we arrive at

$$\Delta l = \frac{2\pi R_0^2 IB}{R_0 IB - \pi r^2 E}\,.$$

However, we are interested in the shrinkage of the loop, so for the given values we obtained the result $|\Delta l| \doteq 0.85\,\mathrm{mm}$.

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Problem FoL.42 ... astronaut training — 5 points

There is a swing carousel standing on the Earth's surface (in gravity g). With what frequency should it turn so that the visitors riding it would experience net acceleration $2g$? The carousel may be considered a circle with radius $R = 3.5\,\mathrm{m}$ placed $H = 4\,\mathrm{m}$ above the ground with seats attached to the perimeter of this circle by chains. Let's assume that a person sitting in a seat is a point mass at distance $l = 3\,\mathrm{m}$ from the point where the seat is attached to the circle, chains and seats are massless, chains have constant lengths and able to rotate freely around the points where they're attached to the circle. Neglect resistive forces.

Karel was watching the Kosmo series.

In the frame of a person riding on the carousel, the person experiences a gravitational force and a centrifugal force pushing him/her into the seat. Those forces are balanced by the reaction of the seat. The total acceleration can be obtained by dividing the forces by the person's mass. This acceleration is supposed to be equal to $2g$. The centrifugal acceleration then must be

$$a = \sqrt{(2g)^2 - g^2} = \sqrt{3}g\,.$$

At the same time, it holds

$$a = \omega^2 r = 4\pi^2 f^2 r\,,$$

where r is the radius of the seat moving around the central axis and f is the desired frequency. Radius r is given by the sum of the circle radius R and the length of the horizontal projection of the chain. Now, the horizontal length is to the total length as the centrifugal acceleration is to the total acceleration, i.e. in the ratio $\sqrt{3}/2$. So

$$r = R + \frac{\sqrt{3}}{2}l\,.$$

Substituting into the frequency formula, we get the result

$$f = \frac{1}{2\pi}\sqrt{\frac{a}{r}} = \frac{1}{2\pi}\sqrt{\frac{\sqrt{3}g}{R + \frac{\sqrt{3}}{2}l}} \doteq 0.266\,\mathrm{Hz}\,.$$

The carousel would have to rotate with frequency of 0.266 Hz to produce the desired acceleration.

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Problem FoL.43 ... exotic field 6 points

Vítek found a very exotic field in an old textbook. Consider a point-like mass $m = 1.00\,\mathrm{kg}$ at rest in equilibrium in the potential $V(x) = -Ax^{10}\exp(Bx)$, where $A = 2.00\,\mathrm{kg{\cdot}m^{-8}{\cdot}s^{-2}}$ and $B = 0.01\,\mathrm{m^{-1}}$. The mass is then slightly displaced from its equilibrium position. Find the period of small oscillations of the point-like mass around the equilibrium position in this potential field. *Vítek is imagining how Matěj is conducting experiments.*

To solve this problem we have to realize that every smooth enough function can be approximated at its minimum by a parabola. The stiffness of the system can be replaced with $V''(x_0)$, where x_0 is the equilibrium position. As the first step we find this position,

$$\begin{aligned} V'(x) &= -\left(10 + Bx\right) Ax^9 \exp(Bx)\,, \\ V'(x_0) &= 0\,, \\ x_0 &= -\frac{10}{B}\,. \end{aligned}$$

Now we only have to work out the second derivative of the potential and substitute the equilibrium position we just computed. So

$$\begin{aligned} V''(x) &= -\left(Bx + \left(9 + Bx\right)\left(10 + Bx\right)\right) Ax^8 \exp(Bx)\,, \\ V''(x_0) &= \frac{10^9 A}{B^8}\exp(-10)\,. \end{aligned}$$

The period of small oscillations is then determined as

$$T = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{m}{V''(x_0)}} = 2\pi\sqrt{\frac{mB^8}{10^9 A}\exp(10)}\,.$$

Plugging in the numerical values we get $T \doteq 2.09 \cdot 10^{-10}\,\mathrm{s}$.

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Problem FoL.44 ... identifying a rod

8 points

Vítek had a cylindrical wire with radius $R = 1.20\,\mathrm{cm}$ made from an unknown material connected to a DC source. Luckily, he knew the energy density of magnetic field $w_{\mathrm{mag}} = 0.50\,\mathrm{J{\cdot}m^{-3}}$ and the energy density of electric field $w_{\mathrm{el}} = 2.50\cdot 10^{-17}\,\mathrm{J{\cdot}m^{-3}}$ immediately above the surface of the conductor. Both fields are parallel with the surface of the conductor. Moreover, using a magnet Vítek found out that the rod is non-magnetic. The conductor is placed in air, the Hall effect is ignored. Find the conductivity of the cylinder's material.

Vítek wanted to make a problem that just needs standard formulas.

With the use of Ohm's law we can express the unknown conductivity in terms of electric field E, which accelerates free charge carries, and current density j as

$$\sigma = \frac{j}{E}\,,$$

where the current density is given by

$$j = \frac{I_R}{S}\,.$$

Here $S = \pi R^2$ denotes the cross-section area up to distance R from the centre of the conductor and I_R is the current passing along the axis of the conductor. The energy densities of the electric and magnetic field are defined as

$$w_{\mathrm{el}} = \frac{1}{2}\varepsilon_0 E^2\,,$$

$$w_{\mathrm{mag}} = \frac{1}{2\mu_0}B^2(R)\,.$$

Electric field E is constant, so we need to find only the magnitude of stationary magnetic field $B(R)$. This can be done with the use of Ampère's law,

$$B(R) = \frac{\mu_0 I_R}{2\pi R}\,.$$

Now by substituting into the definition of w_{mag} and working out the electric current, we get

$$I_R^2 = \frac{8\pi^2 R^2 w_{\mathrm{mag}}}{\mu_0}\,.$$

Rearrangement of the formula defining w_{el} leads to

$$E^2 = \frac{2w_{\mathrm{el}}}{\varepsilon_0}\,.$$

Now, we have all ingredients to express the conductivity in known variables as

$$\sigma = \frac{2}{R}\sqrt{\frac{\varepsilon_0}{\mu_0}}\sqrt{\frac{w_{\mathrm{mag}}}{w_{\mathrm{el}}}} = \frac{2}{R}\frac{1}{Z_0}\sqrt{\frac{w_{\mathrm{mag}}}{w_{\mathrm{el}}}}\,,$$

where Z_0 is the impedance of vacuum. For given values this yields $\sigma \doteq 6.26\cdot 10^7\,\Omega^{-1}{\cdot}\mathrm{m}^{-1}$. Consulting the tables we conclude that the conductor is made of silver.

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Problem FoL.45 ... doublets

8 points

We used a sodium lamp to symmetrically illuminate two $b = 1.0\,\mu\mathrm{m}$ *wide slits* $a = 0.10\,\mathrm{mm}$ *apart. The light passing through the slits is observed on a screen at distance* $d = 1.0\,\mathrm{m}$. *Determine the ratio of intensity of incident light in the 1*st *minimum to the intensity in the 0*th *maximum.*

Assume that a sodium lamp emits light only on two wavelengths $\lambda_1 = 589.0\,\mathrm{nm}$ *and* $\lambda_2 = 589.6\,\mathrm{nm}$ *with intensity ratio 2:1.* *Xellos likes to enlighten people.*

It is well-known that the zeroth maximum of an interference pattern lies at the axis between the slits ($y = 0$) and the first minimum, with zero intensity, lies at $y = d\lambda/2a$, where λ is the wavelength of the incident light on the screen. This holds for a single wavelength. In this case, there are two but with similar values, so we can safely assume that the first minimum is at $y = d\lambda_1/2a + \Delta y$ where $\Delta y \ll d\lambda_1/2a$.

The intensity on the screen from one wavelength λ up to this first minimum is given approximately as (using $\mathrm{sinc}(x) \equiv \sin x/x$ where I_0 is the intensity at the zeroth maximum)

$$I = I_0 \cos^2\left(\frac{\pi a y}{\lambda d}\right) \mathrm{sinc}^2\left(\frac{\pi b y}{\lambda d}\right) .$$

The second term can be neglected, since $b \ll a$ and therefore the argument of sinc is almost zero at the first maximum, i.e. sinc is approximately equal to one. In the vicinity of the first minimum and for $\lambda \approx \lambda_1$ we can make an estimate

$$\cos\frac{\pi a y}{\lambda d} = \cos\left(\frac{\pi a}{\lambda d}\left(\frac{d\lambda_1}{2a} + \Delta y\right)\right) = \sin\left(\frac{\pi}{2}\frac{\lambda - \lambda_1}{\lambda} - \frac{\pi a}{\lambda d}\Delta y\right) \approx \frac{\pi}{2}\frac{\lambda - \lambda_1}{\lambda} - \frac{\pi a}{\lambda d}\Delta y \,,$$

$$I \approx I_0 \left(\frac{\pi}{2}\frac{\lambda - \lambda_1}{\lambda} - \frac{\pi a}{\lambda d}\Delta y\right)^2 .$$

For two wavelengths, the total intensity is $I = I_1 + I_2$, where intensity corresponding to the first wavelength (at the zeroth maximum) is $2I_0$ and I_0 is for the second. Using the approximation above, we have for the first minimum

$$\begin{aligned}\frac{I}{I_0} &= 2\cos^2\left(\frac{\pi a y}{\lambda_1 d}\right)\mathrm{sinc}^2\left(\frac{\pi b y}{\lambda_1 d}\right) + \cos^2\left(\frac{\pi a y}{\lambda_2 d}\right)\mathrm{sinc}^2\left(\frac{\pi b y}{\lambda_2 d}\right) \\ &\approx 2\left(\frac{\pi a}{\lambda_2 d}\Delta y\right)^2 + \left(\frac{\pi}{2}\frac{\lambda_2 - \lambda_1}{\lambda_2} - \frac{\pi a}{\lambda_2 d}\Delta y\right)^2 \\ &\approx 2\left(\frac{\pi a}{\lambda d}\Delta y\right)^2 + \left(\frac{\pi}{2}\delta\lambda - \frac{\pi a}{\lambda d}\Delta y\right)^2 ,\end{aligned}$$

where we denoted one of the wavelengths as λ and introduced the difference $\delta\lambda = |\lambda_1 - \lambda_2|/\lambda$; because the values of wavelengths are similar, the result should depend only on their difference and approximate values (meaning that we can substitute either of the two values for λ). We obtained a quadratic expression for I/I_0 in the form $2A^2x^2 + (C - Ax)^2 = 3A^2x^2 + C^2 - 2ACx$, where unknown x corresponds to Δy and the coefficients are defined as $C = \pi\delta\lambda/2$ and $A = \pi a/\lambda d$. It is easy to show that the minimum of such expression is at $x = \frac{C}{3A}$ and attains the value of $2C^2/3$. The intensity of light in the first minimum is

$$I_{\mathrm{min},\,1} = I_0 \frac{2}{3}\left(\frac{\pi}{2}\delta\lambda\right)^2 ,$$

which must be divided by the reference intensity (at the zeroth maximum) $2I_0 + I_0 = 3I_0$. The resulting ratio is

$$\frac{I_{\text{min, 1}}}{I_{\text{max, 0}}} = \frac{2}{9}\left(\frac{\pi}{2}\delta\lambda\right)^2 \doteq 5.7 \cdot 10^{-7}\,.$$

We may notice that the result is heavily dependent on the intensity ration of the two sodium lines, but it is completely independent of the geometry of the experiment!

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Problem FoL.46 ... stellar sail 8 points

At a distance $d = 100\,\mathrm{AU}$ *from the Sun, there is a mirror with area* $S = 100\,\mathrm{km}^2$ *moving away from the Sun with velocity* $v = 0.4c$. *The mirror is quite strange – it completely reflects light in the range* $\langle 400\,\mathrm{nm}, 500\,\mathrm{nm}\rangle$ *and is completely transparent for all other wavelengths. Find the force due to solar radiation acting on the mirror, according to an observer at the mirror. Assume that the Sun is an ideal black body with surface temperature* $T = 6000\,\mathrm{K}$ *and radius* $R = 7 \cdot 10^5\,\mathrm{km}$. *Xellos would like to have a Solar Sailer.*

The radiation intensity curve of a black body is described by Planck's law

$$\Delta\Phi(\nu) = \frac{2h\nu^3}{c^2}\frac{1}{\exp(h\nu/k_B T) - 1}\Delta\nu\,,$$
$$\Delta\Phi(\lambda) = \frac{2hc^2}{\lambda^5}\frac{1}{\exp(hc/\lambda k_B T) - 1}\Delta\lambda\,,$$

where $\Delta\lambda$ is a small wavelength interval centered around wavelength λ (similarly for frequency $\nu = c/\lambda$). $\Delta\Phi$ is the emitted flux per unit solid angle on this wavelength interval; The solid angle covered by the mirror with area S is $\vartheta \approx S/d^2$ and the incident power (in the given wavelength interval) is given as $\Delta P = \Delta\Phi \cdot \vartheta$. Since the momentum of a photon with relativistic mass m is mc and its energy is mc^2, we can transform the power (energy per time) to force (momentum per time) by diving it by c. The mirror doesn't absorb but reflects the photons, so the relation between incident power P and the force in question (denoted by F) is

$$F = \frac{2P}{c}\,.$$

Now we need to determine P. Since we know the dependence of P on λ, we can determine the total power in the interval $\langle 400\,\mathrm{nm}, 500\,\mathrm{nm}\rangle$ by numerical integration (that is, by division into subintervals and summing the partial powers). But there is a catch: the Planck's law depends on the frame of reference. We will continue to work in an inertial frame with velocity v (slow acceleration during a short time interval can be neglected). The formula for $\Delta\Phi$ then changes to

$$\Delta\Phi'(\nu') = \frac{2h\nu'^3}{c^2}\beta^3\frac{1}{\exp(h\nu'/k_B\beta T) - 1}\Delta\nu'\,.$$

The frequency in mirror's frame of reference is denoted as ν'. We also introduced

$$\beta = \sqrt{\frac{1 - v/c}{1 + v/c}}\,.$$

The relativistic Doppler effect provides us with $\nu' = \nu\beta$ and we can see that the expression for $\Delta\Phi'(\nu)$ looks almost as if we just substituted ν, only that the factor is β^3 instead of β^{-4}. The derivation of $\Delta\Phi'$ can get complicated, so let us only remark that the result must describe a black body radiation curve, but for a different temperature, $T' = \beta T$, and that the factor β^3 comes from spacetime deformation. Using the wavelength λ' in the mirror's frame of reference (keeping in mind that the reflectance is given in this frame) we can write

$$\Delta\Phi'(\lambda') = \frac{2hc^2}{\lambda'^5}\beta^3 \frac{1}{\exp(hc/\lambda' k_B \beta T) - 1}\Delta\lambda' \,.$$

The following conversion to power is also tricky, because from the reference frame of the mirror the Sun is at distance $d' = d\gamma$, where $\gamma = 1/\sqrt{1 - v^2/c^2}$. The angle under which we observe the solar radiation is then equal to $\vartheta' = \vartheta/\gamma^2 = \vartheta(1 - v^2/c^2)$. In conclusion, the radiation force is

$$F = \frac{4hSc}{d^2}\left(1 - v/c\right)^{5/2}\left(1 + v/c\right)^{-1/2} \int_{\lambda_b}^{\lambda_r} \lambda'^{-5} \frac{1}{\exp\left(\sqrt{\frac{c+v}{c-v}}\frac{hc}{\lambda' k_B T}\right) - 1} \mathrm{d}\lambda' \doteq 1.32 \cdot 10^{-22}\,\mathrm{N} \,.$$

This result can be obtained in another way while keeping the required precision. Instead of the integration, we evaluate the integrand at $\lambda' = 450\,\mathrm{nm}$ and multiply it by the length of the interval $100\,\mathrm{nm}$ (a midpoint-rectangular rule). The result will differ only by circa $0.02\,\%$.

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Problem FoL.47 ... throw over a cylinder 6 points

There is a huge cylinder with base radius $R = 10\,\mathrm{m}$ lying on its side. You want to throw a stone from ground level so that the stone flies over the cylinder. What is the minimum required initial velocity of the stone? *Jáchym guessed the original task from a figure in one spanish book.*

We choose the origin of our Cartesian coordinate system so that the cylinder is described by

$$x^2 + (y - R)^2 = R^2 \,.$$

The stone we are throwing travels along a parabola

$$y = -ax^2 + c \,, \tag{2}$$

where a, c are unknown, positive constants. The linear term is missing since the parabola must be symmetrical about the vertical axis, which follows from the nature of the problem. We need to fit the circle under the parabola. To minimize the initial velocity, the circle and the parabola must be as close as possible – the touch in two points symmetric about the y axis (there could be possibly just one point of contact at the top of the cylinder, but this option will be ruled out by the following solution). The crossing of the curves can be obtained by substituting the second equation into the first one,

$$a^2x^4 + \left(1 + 2a\left(R - c\right)\right)x^2 + c\left(c - 2R\right) = 0 \,,$$

and solving this biquadratic equation for x:

$$x^2 = \frac{2a(c-R) - 1 \pm \sqrt{(1+2a(R-c))^2 + 4a^2c(2R-c)}}{2a^2}\,.$$

We want the curves to be tangent and there can be no more than two touching points, one positioned at $-x_\mathrm{t}$ and the second one at x_t. Therefore, only one x_t^2 exists, and so the biquadratic equation must have a zero discriminant. This means that

$$4R^2a^2 + 4(R-c)a + 1 = 0$$

and subsequently

$$c = \frac{4R^2a^2 + 4Ra + 1}{4a}\,. \tag{3}$$

Let's denote the intersection of the parabola and the x axis (the initial coordinate of the stone) by $-x_0$, then the impact point lies at x_0. For the time of travel T, it holds

$$T = \frac{2v_y}{g} = \frac{2x_0}{v_x}\,,$$

which can be rewritten into

$$x_0 = \frac{v_x v_y}{g}\,. \tag{4}$$

The position of the stone in time is described by

$$x = -x_0 + v_x t\,,$$
$$y = v_y t - \frac{1}{2}gt^2\,.$$

We solve the first equation for t, insert it into the second one and get the formula for the path of projectile

$$y = \frac{v_y}{v_x}\left(x_0 + x\right) - \frac{g}{2v_x^2}\left(x_0 + x\right)^2\,.$$

Substituting for x_0 into the equation (4) leads to

$$y = -\frac{g}{2v_x^2}x^2 + \frac{v_y^2}{2g}\,,$$

which is similar in form to the expression (2). By comparison of these two we get

$$a = \frac{g}{2v_x^2}\,,$$
$$c = \frac{v_y^2}{2g}\,.$$

The initial velocity of the stone is then given by

$$v^2 = v_x^2 + v_y^2 = g\left(\frac{1}{2a} + 2c\right)\,.$$

Substituting for c from expression (3), we obtain

$$v^2 = g\left(2R^2a + 2R + \frac{1}{a}\right)\,.$$

The velocity magnitude v is minimal when v^2 is. The minimum of v^2 is obtained by taking its derivative with respect to a and equating it to zero:

$$\frac{\mathrm{d}v^2}{\mathrm{d}a} = g\left(2R^2 - a^{-2}\right) = 0\,,$$
$$a = 2^{-1/2}R^{-1}\,.$$

Finally, we arrive at the numerical value of the initial velocity $v = \sqrt{2gR(1+\sqrt{2})} \doteq 21.8\,\mathrm{m\cdot s^{-1}}$.

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Problem FoL.48 ... evil spring 9 points

Consider a homogeneous spring with rest length $l = 11.3\,\mathrm{cm}$, stiffness $k = 125\,\mathrm{kg\cdot s^{-2}}$ and linear density $\lambda = 9.7\,\mathrm{kg\cdot m^{-1}}$ placed on a horizontal pad. We fix one of its endpoints in place in such a way that the spring can still freely rotate around this endpoint. Then, we remove the pad. Calculate the difference between the total potential energy of the spring in its new equilibrium position and in the initial position. *Jáchym couldn't figure out the principle of a pen.*

A spring hanging freely in a gravitational field extends more close to the point of suspension than at the bottom. So, the linear density is not a constant anymore. First let us find appropriate coordinates to deal with this problem.

Assume an element of length Δx positioned at point x measured from the point of suspension. Below this elements hangs mass $m(x) = (l-x)\lambda$, so there is downward force $m(x)g$ acting on this element. In consequence the element will extend to

$$\Delta y = \Delta x + \frac{m(x)g}{k'} = \Delta x\left(1 + \frac{(l-x)\lambda g}{kl}\right)\,, \tag{5}$$

where k' is stiffness of the chosen element. This stiffness is proportional to the length of the elements (the reasoning being based on the properties of tension in the spring), that is

$$k' = k\frac{l}{\Delta x}\,.$$

The appropriate coordinate for the elongated state is

$$y = \int_0^x \mathrm{d}y = x + \int_0^x \frac{m(x)g}{k'}\,\mathrm{d}x = x + \frac{g\lambda}{kl}\int_0^x (l-x)\,\mathrm{d}x = \left(1 + \frac{g\lambda}{k}\right)x - \frac{g\lambda}{2kl}x^2\,.$$

we used the fact that $\Delta y/\Delta x$ from equation (5) converges to the derivative $\mathrm{d}y/\mathrm{d}x$ in the zero length limit.

Now let's compute the change in potential energy. In the initial state all points are at height 0, so $E_{g_0} = 0$. After reaching the equilibrium each element $\mathrm{d}x$ with mass $\mathrm{d}m = \lambda\mathrm{d}x$ is in height $-y$. It holds

$$E_g = \int_0^m g\,(-y)\,\mathrm{d}m = -g\lambda\int_0^l \left(\left(1+\frac{g\lambda}{k}\right)x - \frac{g\lambda}{2kl}x^2\right)\mathrm{d}x = -\left(1 + \frac{2g\lambda}{3k}\right)\frac{g\lambda l^2}{2}\,.$$

Next we have to determine the change in elastic potential energy. Initially, $E_{\mathrm{p}0} = 0$. In the final state each element Δx extends to Δy, which corresponds to energy change

$$\Delta E_{\mathrm{p}} = \frac{1}{2}k' \left(\Delta y - \Delta x\right)^2 = \frac{kl\left(\Delta y - \Delta x\right)^2}{2\Delta x} = \frac{\left(l - x\right)^2 \lambda^2 g^2}{2kl}\Delta x\,.$$

The expression $\Delta E_{\mathrm{p}}/\Delta x$ can be considered as the linear energy density (per unit length of the initial state), which can be integrate to

$$E_{\mathrm{p}} = \int_0^l \mathrm{d}E_{\mathrm{p}} = \frac{g^2\lambda^2}{2kl}\int_0^l \left(l - x\right)^2 \,\mathrm{d}x = \frac{g^2\lambda^2}{2kl}\left[l^2x - lx^2 + \frac{x^3}{3}\right]_0^l = \frac{g^2\lambda^2 l^2}{6k}\,.$$

Finally we determine the total change in potential energy

$$\Delta E = \Delta E_g + \Delta E_{\mathrm{p}} = E_g - E_{g0} + E_{\mathrm{p}} - E_{\mathrm{p}0} = -\left(1 + \frac{g\lambda}{3k}\right)\frac{g\lambda l^2}{2} \doteq -0.762\,\mathrm{J}\,.$$

At the end let us make a short comment. It is no coincidence that the term

$$-\frac{g\lambda l^2}{2}$$

represents the change of potential energy in the case of zero elongation. Taking the limit of infinite stiffness, $k \to \infty$, we obtain exactly this term. The other term then represents the energy coming from the elongation. We can notice that the released gravitational potential energy was two times as large as the elastic energy stored in the elongated spring.

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Problem M.1 ... Dyson — 3 points

A Dyson sphere is a (so far) hypothetical construction around a star which completely surrounds the star in order to capture all the energy it radiates. Imagine that someone takes our Earth and creates a Dyson sphere from its matter, with our Sun at the centre of the sphere. The outer radius of this sphere is the same as the average Earth-Sun distance $d_{\mathrm{S}} = 149\,600\,000\,\mathrm{km}$. *Assume that Earth is a sphere with radius* $R_{\mathrm{Z}} = 6\,378\,\mathrm{km}$ *and that its matter is homogeneous and incompressible – therefore, the density of the body of the Dyson sphere is the same as the density of Earth. What is the thickness of this Dyson sphere?*

Karel likes problems with a Dyson sphere.

Let's use the fact that volume is conserved. The volume of Earth is $V_{\mathrm{Z}} = \frac{4}{3}\pi R_{\mathrm{Z}}^3$. To calculate the volume of the Dyson sphere, we can use the approximation that the thickness h of the sphere is much smaller than its radius. Therefore, we may calculate its volume as the product of its surface area (either inner or outer, it does not matter in this approximation) and thickness

$$V_{\mathrm{D}} \approx S_{\mathrm{D}}h = 4\pi d_{\mathrm{S}}^2 h\,.$$

You can prove by yourself that if we calculated this volume as a difference of volumes of two spheres $V'_\mathrm{D} = \frac{4}{3}\pi d_\mathrm{S}^3 - \frac{4}{3}\pi(d_\mathrm{S} - h)^3$, we would get an almost identical result. Now, we simply use the equation $V_\mathrm{Z} = V_\mathrm{D}$ to get

$$\frac{4}{3}\pi R_\mathrm{Z}^3 = 4\pi d_\mathrm{S}^2 h\,,$$

$$h = \frac{R_\mathrm{Z}^3}{3d_\mathrm{S}^2} \doteq 3.86 \cdot 10^{-3}\,\mathrm{m}\,.$$

We can see that the thickness is in the order of milimetres, fourteen orders of magnitude smaller than d_S. Therefore, we may conclude that our approximation is valid.

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Problem M.2 ... Dyson Reloaded

3 points

Let's consider the Dyson sphere from the previous problem. (Imagine that someone takes our Earth and creates a Dyson sphere from its matter, with our Sun at the centre of the sphere. The outer radius of this sphere is the same as the average Earth-Sun distance $d_\mathrm{S} = 149\,600\,000\,\mathrm{km}$. Assume that Earth is a sphere with radius $R_\mathrm{Z} = 6\,378\,\mathrm{km}$ and that its matter is homogeneous and incompressible – therefore, the density of the body of the Dyson sphere is the same as the density of Earth.) Calculate the gravitational acceleration that could be felt by astronauts standing on its outer surface. The mass of Earth is $M_\mathrm{E} = 5.97 \cdot 10^{24}\,\mathrm{kg}$ and the mass of the Sun is $M_\mathrm{S} = 1.99 \cdot 10^{30}\,\mathrm{kg}$. *Karel really likes problems with a Dyson sphere.*

To tackle this problem, we will use the Gauss's law for gravity, which states that the gravitational flux through any closed surface is proportional to the enclosed mass. This implies that the gravitational force is going to be the same as if there was a point-like mass $M_\mathrm{Z} + M_\mathrm{S}$ in the middle of the sphere. Since the gravitational force of the Dyson sphere is way smaller than the gravitational force of the Sun, we can neglect it. Using the Newton's law of universal gravitation, one obtains

$$a_g = \frac{GM_\mathrm{S}}{d_\mathrm{S}^2} \doteq 5.93 \cdot 10^{-3}\,\mathrm{m{\cdot}s^{-2}}\,,$$

where G is the Gravitational constant.

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Problem M.3 ... Dyson Reloaded Reloaded

3 points

For now, let's keep discussing the same Dyson sphere. (Imagine that someone takes our Earth and creates a Dyson sphere from its matter, with our Sun at the centre of the sphere. The outer radius of this sphere is the same as the average Earth-Sun distance $d_\mathrm{S} = 149\,600\,000\,\mathrm{km}$. Assume that Earth is a sphere with radius $R_\mathrm{Z} = 6\,378\,\mathrm{km}$ and that its matter is homogeneous and incompressible – therefore, the density of the body of the Dyson sphere is the same as the density of Earth.) We already know that it would be hard to walk on its outer surface, but could we walk on its inner surface? How many days would a month (i.e. 1/12-th of one orbit around the Sun) take, if the sphere rotated with such an angular velocity that the acceleration *due to gravity on its equator would be the same as on the equator of the (regular) Earth?*

Karel really likes problems with a Dyson sphere.

Firstly, we must realize that the gravitational force of the Sun (calculated in the previous problem) is very small, compared to the gravitational acceleration $g = 9.81\,\mathrm{m\cdot s^{-1}}$ certainly negligible. Also, the gravitational force inside of the sphere is zero (result of the Gauss's law). Therefore, we only need to use the formula for centrifugal acceleration

$$g \approx \omega^2 d_\mathrm{S}\,,$$

$$\omega \approx \sqrt{\frac{g}{d_\mathrm{S}}}\,,$$

$$t \approx \frac{\pi}{6}\sqrt{\frac{d_\mathrm{S}}{g}} \doteq 64{,}660\,\mathrm{s} \doteq 0.748\,\mathrm{dne}\,,$$

where we used $t = \frac{T}{12} = \frac{2\pi}{12\omega}$ to calculate the period from the angular velocity.

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Problem M.4 ... Dyson Reloaded[3] 3 points

Let's trouble our minds with the Dyson sphere one last time. I promise it's the last time! (Imagine that someone takes our Earth and creates a Dyson sphere from its matter. The outer radius of this sphere is the same as the average Earth-Sun distance $d_\mathrm{S} = 149\,600\,000\,\mathrm{km}$. Assume that Earth is a sphere with radius $R_\mathrm{Z} = 6\,378\,\mathrm{km}$ and that its matter is homogeneous and incompressible – therefore, the density of the body of the Dyson sphere is the same as the density of Earth.) In this case, let's assume that the engineers made a mistake somewhere and did not build the sphere around the Sun. The sphere stays the same (it has the same radius and it is made of the Earth's matter), it does not rotate and there are no stars or planets on the inside or anywhere nearby. What is the minimum amount of pressure (critical stress) the material of Earth must be able to withstand so that the sphere does not collapse under its own weight? *Matěj became infected and now he also likes Dyson spheres.*

Let's consider a layer of width h, from which we cut out a little part. Now we calculate the force, by which the little part of area $\mathrm{d}S$ (in the little-part approximaion it has the same outer and inner area) is attracted to the centre. Using Gauss's law we get the relation of gravitational acceleration and location in the layer. Let's consider constant density and denote in this case the inner radius as d_S, then

$$a(x) = G\frac{M_\mathrm{Z}x}{h(d_\mathrm{S}+x)^2} \approx \frac{GM_\mathrm{Z}x}{hd_\mathrm{S}^2}\,,$$

where x is the distance of the point of acceleration a from the inner surface of the sphere and $\frac{M_\mathrm{Z}x}{h}$ is (for linear approximation) the mass closer to the centre.[2]

[2] That is basically the same approximation as we used in the M1 problem, calculating the volume, since the inner mass is $\frac{V_\mathrm{in}}{V}M_\mathrm{Z} = \frac{\frac{4}{3}\pi(d_\mathrm{S}+x)^3 - \frac{4}{3}\pi d_\mathrm{S}^3}{\frac{4}{3}\pi(d_\mathrm{S}+h)^3 - \frac{4}{3}\pi d_\mathrm{S}^3}M_\mathrm{Z} \approx \frac{4\pi d_\mathrm{S} x}{4\pi d_\mathrm{S} h}M_\mathrm{Z} = \frac{x}{h}M_\mathrm{Z}$.

Now, integrating the gravitational force we get the force acting upon this part of the layer

$$\mathrm{d}F = \int\limits_0^h a(x)\varrho\,\mathrm{d}S\mathrm{d}x = \frac{GM_\mathrm{Z}\varrho\mathrm{d}S}{hd_\mathrm{S}^2}\int\limits_0^h x\,\mathrm{d}x = \frac{GM_\mathrm{Z}h\varrho\mathrm{d}S}{2d_\mathrm{S}^2} = \frac{GM_\mathrm{Z}\sigma\mathrm{d}S}{2d_\mathrm{S}^2}\,,$$

where ϱ is density and $\sigma = \varrho h = \frac{M_\mathrm{Z}}{4\pi d_\mathrm{S}^2}$ the area density of the layer.

Using analogy of an air bubble on the water surface, this force per unit volume corresponds to pressure

$$\frac{\mathrm{d}F}{\mathrm{d}S} = p_p = \frac{GM_\mathrm{Z}\sigma}{2d_\mathrm{S}^2}\,.$$

Hence, we can imagine the spheric layer as if it was in a medium of constant hydrostatic pressure p_p, which replaces gravitational forces.

Upon its cross-section acts a force

$$F_p = \pi d_\mathrm{S}^2 p_p = \frac{\pi GM_\mathrm{Z}\sigma}{2} = \frac{GM_\mathrm{Z}^2}{8d_\mathrm{S}^2}\,,$$

This is the force, which pushes two hemispheres together. That force spreads over the area of the cross-section (the area is $2\pi d_\mathrm{S}h$). The material, which the layer is made of must therefore withstand pressure

$$p = \frac{F_p}{2\pi d_\mathrm{S}h} = \frac{GM_\mathrm{Z}^2}{16\pi d_\mathrm{S}^3 h} = \frac{3GM_\mathrm{Z}^2}{16\pi d_\mathrm{S} R_\mathrm{Z}^3} \doteq 3.66\,\mathrm{MPa}\,.$$

Because the layer is very thin, we can consider the pressure inside the material constant.

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Problem E.1 ... oh, these batteries — 3 points

Imagine that you have a fully discharged car battery ($U = 12\,\mathrm{V}$, capacity $60\,\mathrm{Ah}$) and a charging device that can supply the battery with a steady current $I = 3.0\,\mathrm{A}$. If you can charge the battery for $t = 75\,\mathrm{min}$, what will its final charge level be (in percent of total capacity)?

Karel. Don't even ask...

Since the value for the battery capacity is in the units of Ah, let's first convert the time to hours, i. e. $t = 1.25\,\mathrm{h}$. In that time, the device transfers to the battery the charge $Q = It = 3.75\,\mathrm{Ah}$. The maximum capacity of the battery is $Q_\mathrm{max} = 60\,\mathrm{Ah}$, so the percentage reached is $k = Q/Q_\mathrm{max} = 6.25\,\%$. The battery will be charged only to $6.3\,\%$.

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Problem E.2 ... inner resistance — 3 points

If we connect two identical resistors in series to a non-ideal DC voltage source, the efficiency of the source is 0.87. What is the efficiency of the source if we connect the two resistors to it in parallel?

We struggled with a revolution.

Let's denote the inner resistance of the source by R_i and the total resistance of the two resistors in series by R. Then, the efficiency of the source can be expressed as

$$\eta_1 = \frac{R}{R + R_\mathrm{i}}\,.$$

It follows from this equation that

$$R_\mathrm{i} = R\frac{1-\eta_1}{\eta_1}\,.$$

And the efficiency with the parallel connection schema (where the total resistance of the two resistors is $R/4$) is

$$\eta_2 = \frac{\frac{R}{4}}{\frac{R}{4} + R_\mathrm{i}} = \frac{R}{R + 4R_\mathrm{i}} = \frac{R}{R + 4R\frac{1-\eta_1}{\eta_1}} = \frac{\eta_1}{4 - 3\eta_1} \doteq 0.626\,.$$

The efficiency of the source is 0.626.

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Problem E.3 ... solar panel incident 3 points

A horizontal photovoltaic panel generates power $300\,\mathrm{W}$ *when sunlight is incident on it perpendicularly. What power would it generate at 10:00, on a day when the Sun rises at 6:00 and sets at 18:00? At noon, the angle of incidence of sunlight on the panel is* 20° *from the vertical.*

Eating green wasabi, remembering green energy, improperly writing gerunds.

The solar panel efficiency depends on the amount of sunlight received and that can be calculated through the perpendicular projection of sunlight on the given panel. If the original incidental power delivered by the sun is P_0, at noon it's given by $P_\mathrm{p} = P_0 \cos 20^\circ$. The true noon is at 12 pm so the angle that we see the sun under at 10 am differs by $\frac{2}{24}360^\circ = 30^\circ$. This difference is in the direction perpendicular to the zenith distance of the Sun at noon (meaning it will not make a difference in that 20 degree angle towards south (or north if we live on the southern hemisphere)) and we simply need to multiply P_p with another projection factor, this time of 30°. The resulting power at 10 am received by the panel is

$$P = P_0 \cos 20^\circ \cos 30^\circ \doteq 244\,\mathrm{W}\,.$$

This computation is so easy because the light rays are perpendicular to Earth's axis on the equinox. In general case, we would use identities of spherical trigonometry.

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Problem E.4 ... transforming 3 points

Consider a transformer that has a primary coil with $N_1 = 100$ *turns and a secondary coil with* $N_2 = 300$ *turns. A light bulb with resistance* $R = 25.0\,\Omega$ *is connected to the secondary coil. We want the light bulb to be supplied with power* $P = 200\,\mathrm{W}$*. What should be the effective voltage connected to the primary coil of the transformer?*

Danka was reminiscing about high school problems.

The effective voltages on the primary and the secondary coil are U_1, U_2, respectively. The power delivered to the bulb as a function of the voltage on it is given by

$$P = \frac{U_2^2}{R}\,.$$

The voltage on the transformer is transferred across with the same ratio as is between the number of loops

$$\frac{U_1}{U_2} = \frac{N_1}{N_2}\,.$$

Now what's left, is to express

$$U_1 = \frac{N_1}{N_2} U_2 = \frac{N_1}{N_2}\sqrt{PR} \doteq 23.6\,\mathrm{V}\,.$$

We need to induce a voltage of 23.6 V on the ends of the primary coil.

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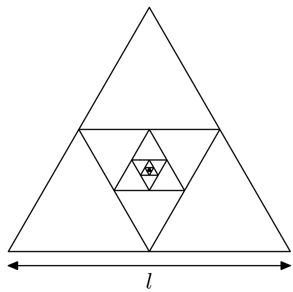
Problem X.1 ... purely fractal

3 points

Matěj likes fractals, so he made one out of infinitely thin wire with linear density $\lambda = 100\,\mathrm{g{\cdot}m^{-1}}$. He started by making an equilateral triangle with side length $l = 10\,\mathrm{cm}$, added the midsegments of this triangle, then added the midsegments of the medial triangle created in the previous step and so on. The result is in figure. What is the total mass of Matěj's fractal?

Matěj likes infinities.



As the midsegment is a half of the length of the side of its original triangle, each added triangle has half the length of a circumference than the previous one. The circumference of the first triangle is $3l$, so the total length of the wire used is

$$3l + \frac{3l}{2} + \frac{3l}{4} + \frac{3l}{8} + \cdots = 3l\sum_{n=0}^{\infty}\frac{1}{2^n} = 6l$$

The total mass is consequently $6l\lambda = 60\,\mathrm{g}$.

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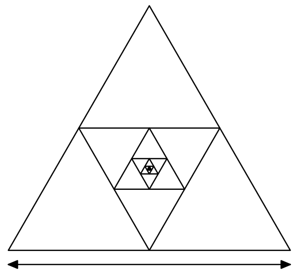
Problem X.2 ... just fractal

3 points

Matěj is still playing with the same fractal (with dimension $l = 10\,\mathrm{cm}$ and linear density $\lambda = 100\,\mathrm{g\cdot m^{-1}}$). He would like to know its moment of inertia with respect to an axis that contains one of the medians of the largest triangle. Find this moment of inertia. *Matěj really likes infinities.*



Moment of inertia of a single equilateral triangle relative to it's median (also its altitude) can be calculated using the formula for the moment of inertia of a homogeneous rod of mass $3l\lambda$ and length $3l$ around the perpendicular axis going through the rod's centre.[3]

The moment of inertia of the largest triangle with respect to the given axis is

$$J_0 = \frac{1}{12} 3l\lambda l^2 = \frac{1}{4}\lambda l^3 \,.$$

Since every additional triangle has sides of half the length, we can express the moment of inertia for the nth triangle as

$$J_n = \frac{1}{4}\lambda \frac{l^3}{\left(2^n\right)^3} = \frac{1}{4}\lambda \frac{l^3}{8^n} \,.$$

The altitudes of all triangles lie on the same axis, hence we can use the additive property of moments of inertia and the total moment can be written as the sum

$$J = \sum_{n=0}^{\infty} J_n = \frac{\lambda l^3}{4} \sum_{n=0}^{\infty} \left(\frac{1}{8}\right)^n = \frac{2\lambda l^3}{7} \doteq 2.9 \cdot 10^{-5}\,\mathrm{kg\cdot m^2}$$

which was evaluated using the formula for an infinite sum of geometric series $\sum_{n=0}^{\infty} a^n = 1/(1-a)$ for $|a| < 1$. We can avoid using the explicit formula through expressing the total moment of inertia as a sum of the moment of the largest triangle and the rest of the fractal (which is half the size and its moment of inertia is $J/8$, as the orientation doesn't matter), which leads to the same result.

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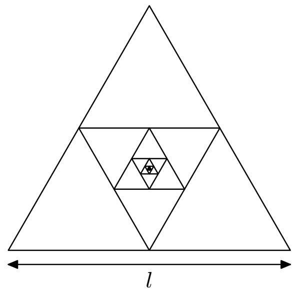
[3] That's because the moment of inertia depends only on the distance of mass from the axis and the triangle has a constant amount of mass at each distance from the axis between 0 and $l/2$

Problem X.3 ... quite fractal
3 points

Now, Matěj hanged his fractal (with dimension $l = 10\,\mathrm{cm}$ and linear density $\lambda = 100\,\mathrm{g{\cdot}m^{-1}}$) from one of the vertices of the largest triangle and let it perform small oscillations around this vertex in the plane that contains the fractal. What period of oscillations did Matěj measure?

Matěj really truly very much does like infinities.



We approach this similarly to the previous problem. First, we calculate the moment of inertia relative to the axis perpendicular to the triangle going through its centre (centroid) – we can talk about a single centroid, because all the triangles and the whole fractal share this centroid (at two thirds of their altitude). The moment of inertia of the largest triangle J_0 consists of three equal moments of the individual sides. Using the parallel axis theorem we have to add the displacement term

$$m\left(\frac{1}{3}\frac{\sqrt{3}}{2}l\right)^2 = \frac{ml^2}{12}\,,$$

to their centre of mass moment of inertia $ml^2/12$ (where $m = \lambda l$). The moment of inertia of the largest triangle is then

$$J_0 = 3\left(\frac{ml^2}{12} + \frac{ml^2}{12}\right) = \frac{1}{2}ml^2 = \frac{1}{2}\lambda l^3\,.$$

Moment of inertia if the nth triangle can be calculated analogously.

$$J_n = \frac{1}{2}\lambda\left(\frac{l}{2^n}\right)^3\,.$$

The total moment of inertia is

$$J_{\mathrm{clk}} = \sum_{n=0}^{\infty} J_n = \frac{1}{2}\lambda l^3 \sum_{n=0}^{\infty}\left(\frac{1}{2^n}\right)^3 = \frac{4}{7}\lambda l^3\,.$$

We can use the parallel axis theorem again to obtain

$$J = J_{\mathrm{clk}} + M\left(\frac{2}{3}\frac{\sqrt{3}}{2}l\right)^2 = \frac{18}{7}\lambda l^3\,,$$

where $M = 6\lambda l$ is the total mass from the first problem. Now what's left is to plug in this value to the formula for period of a physical pendulum

$$T = 2\pi\sqrt{\frac{J}{Mgr}}\,,$$

where $r = l/\sqrt{3}$ is the distance of the centre of mass from the axis of revolution. We arrive to

$$T = 2\pi\sqrt{\frac{3\sqrt{3}l}{7g}} \doteq 0.547\,\mathrm{s}\,.$$

The period of the swings of the fractal is $0.547\,\mathrm{s}$.

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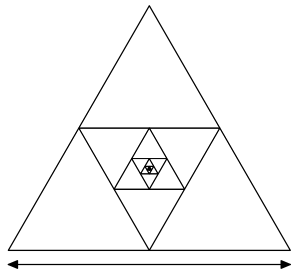
Problem X.4 ... very fractal

3 points



Matěj still likes playing with his fractal (with dimension $l = 10\,\mathrm{cm}$). He found a multimeter and connected it to two vertices of the largest triangle. The resistance of one metre of wire is $1\,000.000\,\Omega$. What resistance did Matěj measure?
Matěj really truly very much does like infinitely infinite infinities.

Let's designate the measured resistance R. In solving this problem, we'll try to make use of the fact that resistance depends on the length of a wire linearly, so if we halve all of the segments of an arbitrary contraption made out of wire, its resistance between arbitrary two points also halves.

If we take away the three outer (largest) sides from Matěj's fractal, we obtain the same fractal just halved in size. Using the statement above, we can claim that the resistance between its vertices is $R/2$. With no effect on the total resistance, we can replace the whole "subfractal" with three resistors of resistance $R/4$ in the star configuration as can be seen in the figure 1.

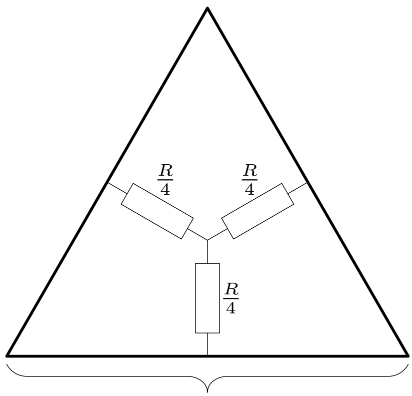


Fig. 1: Replaced subfractal scheme

This is now an easily solvable arrangement of resistors (even symmetrical, so we can ignore the vertical wire of the star). The resistance of half of the outer side is $\lambda l/2$. The total resistance can be expressed as

$$R = \frac{2\frac{\lambda l}{2}\left(2\frac{\lambda l}{2} + \frac{4\frac{R}{4}\frac{\lambda l}{2}}{2\frac{R}{4}+2\frac{\lambda l}{2}}\right)}{2\frac{\lambda l}{2} + 2\frac{\lambda l}{2} + \frac{4\frac{R}{4}\frac{\lambda l}{2}}{2\frac{R}{4}+2\frac{\lambda l}{2}}}$$

After a few algebraic steps we obtain a quadratic equation

$$3R^2 + 2\lambda l R - 2\lambda^2 l^2 = 0$$

which solution is (we are only interested in the positive solution, negative resistance has no meaning in physics)

$$R = \frac{-1 + \sqrt{7}}{3}\lambda l \doteq 54.858{,}4\,\Omega\,.$$

An alternative approach can be developed from the idea that thanks to the symmetry of the fractal the voltage is the same everywhere on the axis of the triangle and consequently we can replace this axis with a single node S. Then we obtain the resistance $R/2$ between the vertex of the fractal and the node S expressed as

$$\frac{2}{R} = \frac{2}{\lambda l} + \cfrac{1}{\cfrac{\lambda l}{2} + \cfrac{1}{\cfrac{2}{\lambda l} + \cfrac{4}{R}}}\,,$$

because the resistance between the vertex of the inner half-fractal and the node S will be $R/4$. This leads to the identical quadratic equation and to the same result.

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