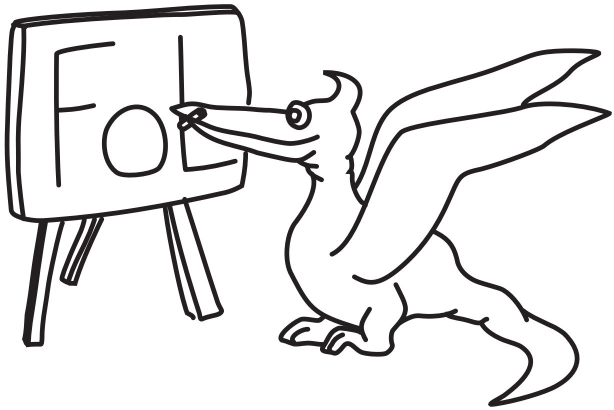
Solutions of 9^{th} Online Physics Brawl



Problem FoL.1 ... inconspicuous escalator 3 points

Suppose that the stairs of an escalator are moving with speed $v = 4.0\,\mathrm{km\cdot h^{-1}}$. How much can the speed of the moving belt the passengers hold onto differ from this speed (in percent), such that the passengers cannot tell the difference between the two speeds? The escalator is $d = 30\,\mathrm{m}$ long and a passengers cannot tell the difference if their hand moves by less than $\Delta x = 5\,\mathrm{cm}$ during the whole journey from one end of the escalator to the other end.

Karel was inspired by Dodo's model, which he then disproved with this problem.

The speed of the stairs is $v = 1.11\,\mathrm{m\cdot s^{-1}}$. A passenger therefore travels from one end of the escalator to the other end in time $t = d/v = 27\,\mathrm{s}$. Hence, the difference in the speed of the belt can be at most $\Delta v = \Delta x/t = \Delta x \cdot v/d \doteq 0.0018\,\mathrm{m\cdot s^{-1}} \doteq 0.0066\,\mathrm{km\cdot h^{-1}}$. The speed of the belt can differ from the speed of the stairs by up to $0.17\,\%$ of the speed of the stairs. Notice that the result is independent of the speed v. This is a consequence of choosing a relative difference as our result.

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Problem FoL.2 ... long day 3 points

Consider a boat floating on the equator. How fast (in knots) does it have to move if one day on the ship should last 25 hours? *A 24-hour day is too short for Matěj.*

The Earth performs one rotation in 24 hours. In 25 Earth days (i.e. time $t = 25\cdot 24\,\mathrm{h}$), passengers onboard should experience only 24 boat days (i.e. $t = 24 \cdot 25\,\mathrm{h}$). This corresponds to exactly one journey around the Earth. Let R be the radius of Earth. Then, the velocity of the boat is $v = 2\pi R/t = 18.5\,\mathrm{m\cdot s^{-1}} = 36\,\mathrm{kt}$.

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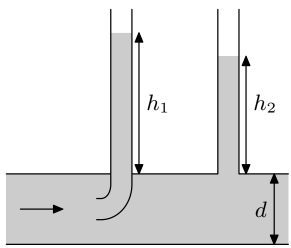
Problem FoL.3 ... pair of pipes 3 points

Water flows without friction in a horizontal pipe with a diameter $d = 10\,\mathrm{cm}$. The pipe is connected to two smaller vertical pipes. The intake of the first pipe faces against the water flow, so it is perpendicular to the horizontal pipe, and the water flowing into it comes to a stop in this pipe. The intake of the second pipe is parallel to the horizontal pipe. The water reaches a height $h_1 = 30.0\,\mathrm{cm}$ in the first smaller pipe, but it only reaches a height $h_2 = 25.0\,\mathrm{cm}$ in the second pipe. What is the velocity of the water flowing in the main pipe?



Jindra was dreaming of water in the summer dry season.

We will start from the Bernoulli equation. All the variables referring to the first smaller pipe are indexed by 1, while variables referring to the second pipe are indexed by 2. The Bernoulli equation states

$$p_1 + \frac{1}{2}\varrho v_1^2 = p_2 + \frac{1}{2}\varrho v_2^2\,.$$

From the problem statement, we deduce that $v_1 = 0$ and v_2 corresponds to the velocity of the water flow in the main tube. The pressures p_1 and p_2 determine the water levels in pipes 1 and 2 respectively.

$$\varrho g h_1 = \varrho g h_2 + \frac{1}{2}\varrho v_2^2$$
$$v_2 = \sqrt{2g(h_2 - h_1)}$$
$$v_2 = 0.99\,\mathrm{m\cdot s^{-1}}$$

The velocity of the water flow is $0.99\,\mathrm{m\cdot s^{-1}}$.

The same principle is used for the design of the Pitot tube, which is used for measuring aircraft speed.

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Problem FoL.4 ... lost heat — 3 points

A calorimeter contains $0.50\,\mathrm{kg}$ of water at a temperature $t_1 = 20\,\mathrm{^\circ C}$. We add another $0.50\,\mathrm{kg}$ of water at a temperature $t_2 = 80\,\mathrm{^\circ C}$ to the calorimeter. After it has reached thermal equilibrium, the water has temperature $t_3 = 45\,\mathrm{^\circ C}$. Determine the heat capacity of the calorimeter. Assume that the calorimeter does not exchange heat with its surroundings.

Jindra was curious where the heat from water in a calorimeter has gone to.

The hot water that is added to the calorimeter transfers heat to both the cold water and the calorimeter. The specific heat of water is $c = 4180\,\mathrm{J\cdot kg^{-1}\cdot K^{-1}}$, and we will assume that it is independent of temperature (in the range from $20\,\mathrm{^\circ C}$ to $80\,\mathrm{^\circ C}$, only the third significant digit of c varies). Let C be the heat capacity of the calorimeter. Then

$$cm_2(t_2 - t_3) = cm_1(t_3 - t_1) + C(t_3 - t_1)\,,$$
$$C = cm_2\frac{t_2 - t_3}{t_3 - t_1} - cm_1\,,$$
$$C \doteq 836\,\mathrm{J\cdot K^{-1}}\,.$$

The heat capacity of the calorimeter is $836\,\mathrm{J\cdot K^{-1}}$.

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Problem FoL.5 ... one hundred — 3 points

Jindra wants to know the amount of heat that needs to be supplied to a cube of ice with mass $1.0\,\mathrm{kg}$ at the temperature $0.0\,\mathrm{^\circ C}$ in order to change it into steam at $100.0\,\mathrm{^\circ C}$. The specific heat of water is $1\,\mathrm{kcal\cdot kg^{-1}\cdot K^{-1}}$; assume that it does not depend on temperature. Look up any other necessary constants.

Jindra ordered boiled ice in the restaurant.

The heat is necessary for the phase change of ice into water, for heating up the water from 0.0 °C to 100.0 °C and for the phase change of water into steam. Therefore, the heat we need to supply is

$$Q = l_\mathrm{t} m + cm\Delta t + l_\mathrm{v} m\,.$$

The latent heat of fusion of water is $3.337 \cdot 10^5\,\mathrm{J{\cdot}kg^{-1}}$ and the latent heat of vaporization is $2.256 \cdot 10^6\,\mathrm{J{\cdot}kg^{-1}}$. A kilocalorie (kcal) is an old unit of energy and it can be converted into joules as $1\,\mathrm{kcal} \doteq 4\,184\,\mathrm{J}$. The heat supplied to the ice turns out to be $Q \doteq 3.01 \cdot 10^3\,\mathrm{kJ}$.

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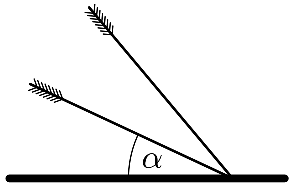
Problem FoL.6 ... archer

4 points

An archer is shooting arrows from a point on a horizontal plane. He manages to land two arrows at essentially the same points, but pointing in different directions. One of the arrows is pointing at an angle $\alpha = 25.00^\circ$ with respect to the horizontal plane. At what angle with respect to the horizontal plane is the second arrow pointing if we assume that both were launched with the same force? *Matěj was shooting arrows.*



It seems that we are not given enough parameters. We have no knowledge of the launch speed nor of the distance of the archer from the point of impact. It will turn out, however, that these are not needed.

The arrow's trajectory can be determined in reverse, starting from the point of impact and launching the arrows with a velocity v. The first arrow is launched at an angle α with respect to the horizontal plane. From the equations of projectile motion, the time of flight is

$$t = \frac{2v\sin\alpha}{g}\,,$$

and the distance at which it lands is

$$s_1 = \cos\alpha v t = \frac{2v^2}{g}\sin\alpha\cos\alpha\,.$$

If we denote the second angle by β, the distance at which the second arrow lands is

$$s_2 = \frac{2v^2}{g}\sin\beta\cos\beta\,.$$

Since the archer launched the arrows from the same place, these distances must be equal

$$s_1 = s_2\,,$$
$$\frac{2v^2}{g}\sin\alpha\cos\alpha = \frac{2v^2}{g}\sin\beta\cos\beta\,,$$
$$\sin\alpha\cos\alpha = \sin\beta\cos\beta\,,$$
$$\sin 2\alpha = \sin 2\beta\,,$$

The last equation has solutions other than $\alpha = \beta$, since the sine is not an injective function (not one-to-one). Considering the nature of archery, we are only looking for solutions within the interval $(0^\circ, 90^\circ)$. We can simplify it to

$$\cos(2\alpha - 90^\circ) = \cos(90^\circ - 2\beta) \,,$$
$$2\alpha - 90^\circ = 90^\circ - 2\beta \,,$$
$$\beta = \frac{1}{2}(180^\circ - 2\alpha) = 90^\circ - \alpha = 65.00^\circ \,.$$

We can see that the orientations of both arrows are symmetric around the angle 45°. It is worth noting that the result is independent of the acceleration due to gravity g, so the same result would be obtained for an archer on a different planet.

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Problem FoL.7 ... strange eclipse 4 points

The maximum duration of a total solar eclipse on the Earth's surface is $t = 449\,\mathrm{s}$. During this time, the stripe of land covered by the total eclipse (the shadow of the Sun) has length $d = 267\,\mathrm{km}$. Find out the maximum duration of a total solar eclipse for a passenger on a plane which flies with a velocity $v = 903\,\mathrm{km/h}$. Neglect the height of the plane above the surface.

Dodo is dreaming about a trip to Argentina.

From the length of the shadow and the duration of the total solar eclipse, we determine the velocity of movement of the shadow on the Earth's surface as $u = d/t$. Then we get the duration of a total solar eclipse T for a passenger on a plane as

$$T = \frac{d}{u - v} = t\frac{d}{d - vt} = t\frac{1}{1 - \frac{vt}{d}} = 777\,\mathrm{s}\,.$$

The passenger will see the total solar eclipse for about 13 minutes.

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Problem FoL.8 ... neglect causality 4 points

Elisa had a daughter Lottie with Johann. Unfortunately, something strange happened to Lottie. She travelled back in time to the past, where she grew up and married Peter. She and Peter had a daughter Elisa, which is also her mother. Therefore, a time paradox is created. The existence of Lottie is caused by the existence of Elisa and vice versa, we can't claim that their existence has a beginning. Assume that each child inherits exactly half of her genetic information from the father and half from the mother. What percentage of Elizabeth's genetic information originally came from Peter? Johann is not related to Peter.

Matěj was watching a series, but I won't reveal its name to avoid spoilers.

Let's denote the genetic information of each family member by the first letter of their name. In accordance with the assumption in the problem statement, we can write

$$A = \frac{1}{2}P + \frac{1}{2}S\,,$$
$$S = \frac{1}{2}J + \frac{1}{2}A\,,$$

which is a system of equations with four unknowns. Then, we can express

$$A = \frac{1}{3}J + \frac{2}{3}P\,.$$

Elisa actually carries 66.7 % of Peter's genetic information.

We have ignored the fact that this whole problem doesn't make sense from a physics point of view, because everything we know so far shows that travelling back in time is impossible in reality.

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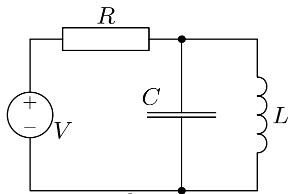
Problem FoL.9 ... fun with RLC — 4 points

Consider the circuit in the figure. It consists of a coil with inductance $L = 10.0\,\mathrm{mH}$, a capacitor with capacitance $C = 4.70\,\mathrm{\mu F}$, a resistor with resistance $R = 1.00\,\mathrm{k\Omega}$ and an ideal source of DC voltage $V = 230\,\mathrm{V}$. Calculate the real electric power consumed by the resistor. The utility frequency in Czech sockets is $f = 50\,\mathrm{Hz}$. *Dodo wants to be cruel.*



The source provides **DC** power, hence the coil would (after the circuit reaches a stationary state) behave as a wire and the capacitor as a "hole" in the circuit. Only the resistor remains connected to the power source. The power consumed by the resistor can be calculated as

$$P = VI = \frac{U^2}{R} = 52.9\,\mathrm{W}\,,$$

which is close to the power of a standard light bulb.

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Problem FoL.10 ... an unfair race — 4 points

Martin and Ivan had a race to the opposite side of a hill. Both of them ran the same horizontal distance $x = 60\,\mathrm{m}$. However, Ivan ran over the hill, which can be described as an arc of a circle with radius $r = 50\,\mathrm{m}$, while Martin ran along a corridor through the hill. The floor of this corridor is an inclined plane and its angle of inclination with respect to the horizontal plane is $\alpha = 14\,^{\circ}$. How much larger (in percent of Martin's average speed) must Ivan's average speed be if both runners reached the end of the corridor (on the opposite side of the hill) simultaneously?

Martin did not feel like running up a hill.

The tunnel itself is a chord of the circular hill. If l is the length of the tunnel, then the central angle φ between the beginning and the end of the tunnel satisfies the equation

$$\sin\left(\frac{\varphi}{2}\right) = \frac{l}{2r}\,.$$

The length of such an arc s is

$$s = r\varphi = 2r \arcsin\left(\frac{l}{2r}\right),$$

while the length of the tunnel is

$$l = \frac{x}{\cos\alpha}\,.$$

The ratio of average speeds of both runners is therefore

$$\frac{v_I}{v_M} = \frac{s}{l} = \frac{2r\cos\alpha}{x} \arcsin\left(\frac{x}{2r\cos\alpha}\right).$$

For the given values, $v_I/v_M \approx 1.078$. Ivan's average speed must be higher by approximately 7.8 %.

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Problem FoL.11 ... Humber Bridge

3 points

The Humber Bridge, which runs across the river Humber in England, is one of the largest suspension bridges in the world. It consists of two 155.5 m *high towers (with height measured from the river surface), whose bases are* 1410 m *away from each other (Euclidean distance, not distance on the ground), connected by ropes that are holding a road. Although both towers are vertical, their highest points are further away from each other than their bases. What is the difference of these (Euclidean) distances?* *Matěj was watching Tom Scott.*

The apparent conflict in the problem statement is caused by curvature of the Earth, which causes vertical objects in different places to be non-parallel. Let us imagine an isosceles triangle whose vertices are the bases of the towers and the middle of the Earth. The lengths of its sides are $s = 1410\,\mathrm{m}$, $R = 6378\,\mathrm{km}$ and R. A second triangle would consist of the towers' highest points and the middle of the Earth. The lengths of its sides are s', $R+h$ a $R+h$, where s' is the distance of towers' highest points and h is the height of a tower. Both triangles are obviously similar, so we can easily calculate

$$s' = s\frac{h+R}{R}\,,$$

$$s' - s = \frac{sh}{R} = 3.44\,\mathrm{cm}\,.$$

The highest points of the towers are therefore more than 3 cm further away from each other than their bases. If we thought of distances measured on a sphere instead, we would (from the similarity of circular sectors) come to the same result.

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Problem FoL.12 ... jumping 4 points

Find the angular velocity necessary to land a qaudruple Axel jump. The figure skater takes off from the left forward outside edge (the outer side of the left-leg ice skate's edge while moving forward) and lands on the right back outside edge, while achieving a maximum height $h = 1.0\,\mathrm{m}$ *above the ice.* *Dodo and the Grand Prix of Figure Skating.*

While jumping to a height h, a figure skater spends a total time t in the air. We can find this time from the equation of motion with constant acceleration (or free fall)

$$h = 1/2g\left(\frac{t}{2}\right)^2 .$$

We can express the time t as

$$t = \sqrt{\frac{8h}{g}} .$$

The skater needs to make four and a half turns, i.e. to rotate by $\varphi = 9\pi$. The angular velocity needed for that can be found as

$$\omega = \frac{\varphi}{t} = 9\pi\sqrt{\frac{g}{8h}} \approx 31.3\,\mathrm{s}^{-1} .$$

As of 27.11.2019, the jump hadn't been performed successfully yet, the only attempt was made by the Russian skater Arthur Dmitriev at Rostelecom Cup.

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Problem FoL.13 ... spider-spider 4 points

Pete the spider is $2\,\mathrm{cm}$ *long and he can weave threads of spider web with different thickness, but he needs to weave a thread whose radius is at least* $50.0\,\mathrm{nm}$ *in order for it to be able to support his own weight. What is the minimum radius of a thread that could support the spider if his size increased to that of a human? Assume that he would be* $2\,\mathrm{m}$ *long, with all his other dimensions increasing accordingly, but the density remaining constant.*
Tomáš was learning mathematics.

Clearly, the mass of the spider increases with the cube of his size (under the assumption that the density is the same). Therefore, after increasing n times in size, the tension in the string increases n^3 times. In our case, $n = 100$. The load capacity of the thread is, however, proportional to the cross-sectional area of the thread, so it increases as the square of the radius. Let m be the ratio of the necessary thread radius to the original thread radius (before the increase in size). Then, we get

$$m^2 = n^3 ,$$

$$m = n^{\frac{3}{2}} = 1000 .$$

The enlarged spider needs to weave a thread with radius at least $50\,\mathrm{\mu m}$. Upon comparison with conventional ropes used to support people, which have radii in the order of centimeters, we can truly appreciate the unbelievable strength of the spider web.

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Problem FoL.14 ... is it colonizable? 4 points

A planet similar to the Earth has a mass $M = 7.166 \cdot 10^{24}\,\mathrm{kg}$*, which is a bit more than the Earth's mass. By coincidence, the first and second cosmic velocities on this planet are the same as on the Earth. What is the gravitational acceleration on its surface?*

Matej can't wait for colonization.

Notice that the first and second cosmic velocities for a given planet depend only on the ratio of its mass and radius. The mass of our planet is $M = 1.20M_\mathrm{Z}$, where M_Z is the mass of the Earth. Analogously, the radius is $R = 1.20R_\mathrm{Z}$ (so that $M/R = M_\mathrm{Z}/R_\mathrm{Z}$). From the formula for gravitational acceleration, it can be seen that we can easily express the result as a multiple of gravitational acceleration on Earth g_Z

$$g = \frac{GM}{R^2} = \frac{1.2GM_\mathrm{Z}}{(1.2)^2 R_\mathrm{Z}^2} = \frac{1}{1.2} g_\mathrm{z} \doteq 0.833\, g_\mathrm{z}\,,$$

where G is the gravitational constant. Although the planet has a bigger mass, it has only 83 % of the Earth's gravitational acceleration.

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Problem FoL.15 ... Dan's fléche burnt. 5 points

Jáchym spilled some lava in Minecraft. Let's assume (in order to simplify) that the world is a 2D cube grid with edge length $1\,\mathrm{m}$*. We can simulate the lava by the following set of rules:*

1. *the lava level in the central cube is* $1\,\mathrm{m}$*,*
2. *the lava level in any block which cannot be reached from the central block by moving through less than 3 other cubes (assume that we can only move between cubes if they have a common face), is* 0*,*
3. *the lava level in other blocks is given by the arithmetic mean of levels in neighbouring blocks (two blocks are neighbours if they have a common face).*

What is the total volume of the spilled lava?

Jáchym poured a bucket of lava on Dan's fléche in Minecraft.

In the picture 1 we can see an outline of part of the situation. As we can see, our problem has several symmetry axes. That makes our work a lot easier. As a result, we only need to find the lava levels in 5 squares – let's denote them a through e. We get a system of linear equations

$$\begin{aligned}
4a &= 1 + 2d + b\,,\\
4b &= a + 2e + c\,,\\
4c &= b\,,\\
4d &= 2a + 2e\,,\\
4e &= d + b\,.
\end{aligned}$$

Its solution is

$$a = \frac{89}{208}\,,$$
$$b = \frac{36}{208}\,,$$
$$c = \frac{9}{208}\,,$$
$$d = \frac{56}{208}\,,$$
$$e = \frac{23}{208}\,.$$

The final volume equals $V = (1 + 4a + 4b + 4c + 4d + 8 \cdot 10^{)}\,\mathrm{m^3} = \frac{72}{13}\,\mathrm{m^3} \doteq 5.54\,\mathrm{m^3}$.

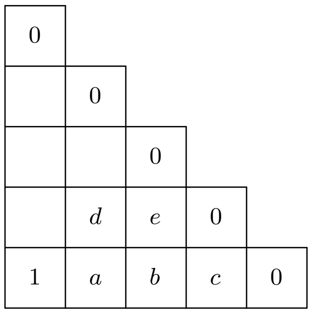

Fig. 1: Outline of part of lava.

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Problem FoL.16 ... one hundred reloaded

4 points

Jindra was interested in the magnitude of error in the problem "one hundred". The specific heat of water is in fact temperature-dependent. Jindra found an empirical relationship for the specific heat $c_{\mathrm{water}} = 3.1832 \cdot 10^{-6} t^4 - 7.7922 \cdot 10^{-4} t^3 + 7.5387 \cdot 10^{-2} t^2 - 2.9190 t + 4.2158 \cdot 10^3$ *where the temperature t must be in* °C *and the specific heat is obtained in* $\mathrm{J \cdot kg^{-1} \cdot K^{-1}}$. *What is the difference (in percent) between the heat supplied to water with a constant specific heat* $c = 4184.0\,\mathrm{J \cdot kg^{-1} \cdot K^{-1}}$ *and the heat calculated using the empirical relationship, if the mass of this water is* $m = 1.00\,\mathrm{kg}$ *and its temperature increases from* $0.00\,°\mathrm{C}$ *to* $100.00\,°\mathrm{C}$ *in both cases? The sign of the result is important (i.e. if the amount of heat for the case with constant specific heat is lower, we expect a negative sign).* *Jindra received steam instead of boiled ice.*

The problem is relatively straightforward, the only tedious part is evaluating the integral. For the constant specific heat

$$Q_1 = cm\Delta t$$
$$Q_1 = 4.1840 \cdot 10^5\,\mathrm{J}\,.$$

For the variable specific heat, we find

$$Q_2 = m \int_0^{100} c(t)\mathrm{d}t$$

$$Q_2 = m \left[\frac{3.1832 \cdot 10^{-6}}{5} \cdot t^5 - \frac{7.7922 \cdot 10^{-4}}{4} \cdot t^4 + \frac{7.5387}{3} \cdot t^3 - \frac{2.9190}{2} \cdot t^2 + 4.2158 \cdot 10^3 t \right]_0^{100}$$

$$Q_2 = 4.1900 \cdot 10^5\ \mathrm{J}\,.$$

The difference (relative to the case with variable specific heat) is

$$\delta Q = \frac{Q_1 - Q_2}{Q_2}$$

$$\delta Q \doteq -0.14\,\%$$

The value $\delta Q = -0.14\,\%$ is small, so the approximation in the problem "one hundred" was justified.

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Problem FoL.17 ... lazy electron — 4 points

Consider the Bohr model of a hydrogen atom – a positively charged nucleus with charge $e = 1.6 \cdot 10^{-19}\ \mathrm{C}$ *and an orbiting electron with the opposite charge and a much smaller mass* $m_e = 9.1 \cdot 10^{-31}\ \mathrm{kg}$. *What would be the radius of the hydrogen atom (in kilometers) if the period of one orbit of the electron took one day? Do not consider the fact that the hydrogen atom could not exist in such a state (or could it...?).*

Matěj was comparing a hydrogen atom with the globe.

In the Bohr model, we can use the classical formulae for the centrifugal force and Coulomb's law and we get

$$\frac{1}{4\pi\varepsilon_0}\frac{e^2}{r^2} = m_e \omega^2 r\,,$$

$$r = \sqrt[3]{\frac{e^2}{4\pi\varepsilon_0 \omega^2 m_e}} = \frac{1}{2\pi}\sqrt[3]{\frac{e^2 T^2}{2\varepsilon_0 m_e}} = 3\,600\ \mathrm{m} = 3.6\ \mathrm{km}\,,$$

where $T = \frac{2\pi}{\omega} = 1\ \mathrm{d} = 86\,400\ \mathrm{s}$ is the period of one orbit, ε_0 is the permittivity of vacuum and r is the atomic radius.

N.B. 1: We assumed that the atomic nucleus is static. Correctly, we should include the motion of the nucleus around the centre of mass of the atom. Then, the distance in Coulomb's law would be the distance from the nucleus instead of the centre of mass

$$r' = r\frac{m_e + M}{M}\,,$$

where M is the nuclear mass. As the nuclear mass is about three orders of magnitude larger than the mass of an electron, the result will change by less than 1% (the correction factor would be present in the result with $\frac{2}{3}$ in the exponent).

N.B. 2: We silently neglected Bohr's quantisation condition $L = n\hbar$ (L is the overall angular momentum, $\hbar$ is the reduced Planck constant and n is a natural number). This condition, however, quantises primarily orbits with small radii (and small angular momenta). When the radius increases, the angular momentum increases as $L = m_e\omega r^2 \approx r^{\frac{1}{2}}$ and for extreme cases such as the one in this problem, the value of L is in the order of millions of $\hbar$, so we can consider L to be continuous.

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Problem FoL.18 ... charge in a box — 4 points

A nasty charge has escaped from an imaginary cylinder, so Jindra locked it into a box. The box has dimensions $8.00 \times 8.00 \times 4.00\,\mathrm{cm}$. *Jindra taped the charge into the middle of its square base. The magnitude of the charge is* $Q = 1.00 \cdot 10^{-6}\,\mathrm{C}$. *Find the electric flux through the opposite face of the box.* *However, electric charges don't like Jindra.*

Let us imagine that we have another box (with the same dimensions) and glue it to the original box such that the face with the charge is their common face. These 2 boxes now form a cube with edge length 8 cm. Therefore, 1/6 of the overall electric flux flows through each face. We calculate the overall flux from Gauss's law.

$$\varphi_{\mathrm{tot}} = \oint_S \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{S} = \frac{Q}{\varepsilon_0}$$

The flux through the face opposite to the glued charge is

$$\varphi_{\mathrm{base}} = \frac{1}{6}\varphi_{\mathrm{tot}} = \frac{1}{6}\frac{Q}{\varepsilon_0}\,,$$
$$\varphi_{\mathrm{base}} = 1.88 \cdot 10^4\,\mathrm{V{\cdot}m}\,.$$

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Problem FoL.19 ... helix — 4 points

A positron is moving at a distance $R = 10\,\mathrm{cm}$ *from a wire carrying a current* $I = 2.5\,\mathrm{A}$. *The current generates a magnetic field, which causes the positron to orbit around the wire. The component of the positron's velocity in the direction of the current is* $v = 1.5\,\mathrm{m{\cdot}s^{-1}}$ *and the radial component is zero. Under the condition that the movement is stable, determine the period of the positron's orbit around the wire (in miliseconds).*
Let $e = 1.6 \cdot 10^{-19}\,\mathrm{C}$, $\mu_0 = 4\pi \cdot 10^{-7}\,\mathrm{N{\cdot}A^{-2}}$, $m_\mathrm{e} = 9.1 \cdot 10^{-31}\,\mathrm{kg}$. *You spin me right round.*

The centripetal force acting on the positron is the Lorentz force

$$\boldsymbol{F} = -e\vec{v} \times \vec{B}\,,$$
$$F = evB(R)$$

where $B(R)$ is the magnitude of the magnetic field at the distance R from the wire, given by Ampére's law

$$B(R) = \frac{\mu_0}{2\pi}\frac{I}{R}$$

The centripetal force determines the angular velocity ω

$$F = m_e\omega^2 R$$

where the period of one orbit T satisfies

$$\omega = \frac{2\pi}{T}$$

Therefore

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m_e R}{F}} = 2\pi\sqrt{\frac{m_e R}{evB(r)}} = 2\pi\sqrt{\frac{2\pi m_e R^2}{ev\mu_0 I}} = 2\pi R\sqrt{\frac{m_e}{\frac{\mu_0}{2\pi}evI}} \approx 1.7\,\mathrm{ms}$$

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Problem FoL.20 ... charge inside a cylinder 6 points

In a thought experiment, Jindra managed to catch an electric charge $Q = 1.00\cdot 10^{-6}\,\mathrm{C}$ *into a cylinder which does not interact with the charge. The charge remains stationary in the centre of the cylinder. The cylinder has radius* $r = 4.00\,\mathrm{cm}$ *and height* $v = 6.00\,\mathrm{cm}$*. Determine the electric flux through the top base of the cylinder.* *Jindra likes electric charges.*

The electric flux is

$$\varphi = \int_S \boldsymbol{E}\cdot \mathrm{d}\boldsymbol{S}\,.$$

Gauss's law states that the electric flux through a closed surface is proportional to the charge enclosed by this surface

$$\varphi = \oint_S \boldsymbol{E}\cdot \mathrm{d}\boldsymbol{S} = \frac{Q}{\varepsilon_0}$$

Imagine the circumscribed sphere around the cylinder (the sphere that contains the perimeters of both bases). The flux through the top base of the cylinder is equal to the flux through the spherical cap over this base. Furthermore, the vector of the electric field is normal to the surface of the sphere and it has a constant magnitude on this surface. The area of the spherical cap is

$$S_{\mathrm{cap}} = 2\pi h R\,,$$

where h is the height of the spherical cap and R is the radius of the sphere, in our case $R = \sqrt{r^2 + (v/2)^2} = 5.00\,\mathrm{cm}$ and $h = R - v/2 = 2.00\,\mathrm{cm}$. The ratio of the electric flux through the spherical cap to the flux through the entire sphere is the same as the ratio of S_{cap} to the surface area of the entire sphere

$$\frac{\varphi_{\mathrm{cap}}}{\varphi_{\mathrm{sphere}}} = \frac{\varphi_{\mathrm{cap}}}{\frac{Q}{\varepsilon_0}} = \frac{S_{\mathrm{cap}}}{S_{\mathrm{sphere}}} = \frac{h}{2R}\,,$$

$$\varphi_{\mathrm{cap}} = \frac{h}{2R}\frac{Q}{\varepsilon_0}\,,$$

$$\varphi_{\mathrm{cap}} = 2.26\cdot 10^4\,\mathrm{V{\cdot}m}\,.$$

The electric flux through the top base of the cylinder is therefore $\varphi_{\mathrm{cap}} = 2.26\cdot 10^4\,\mathrm{V{\cdot}m}$.

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Problem FoL.21 ... African Sun 4 points

Mišo went on another train adventure – he wanted to cross the entire Africa by train. In the middle of September, before arriving to Nairobi, he saw a reflection of the just-rising Sun in the window on the right while sitting in the direction of the route (facing in this direction). At that time, the train was moving exactly south. 10 minutes later, the train was moving south-southwest. Determine the angle (in degrees) by which the reflection of the Sun with respect to the wagon deviated from the original position from Mišo's point of view.

Dodo was going home by a night train.

The train turned by the angle between the south and south-southwestern direction, which is $\alpha = 22.5\,^{\circ}$. During that time, the Sun was also moving on the sky. Taking into account that this event happened near the equator and the date was close to the solar equinox, we could make an approximation that the Sun rose exactly in the East and ascended perpendicularly to the horizon with an angular velocity $\omega = 15\,^{\circ}/\mathrm{h}$. Therefore, after 10 minutes, it rose to the angle $\beta = 2,5^{\circ}$ above the horizon. These two angles are measured in perpendicular directions, so we can use the Pythagorean theorem as an approximation and we get that in total, the Sun turned by the angle

$$\gamma = \sqrt{\beta^2 + \alpha^2} = 22.64\,^{\circ}\,,$$

with respect to the train. The correct solution needs to use the law of cosines for a spherical triangle and then we get $\gamma = \arccos\left(\cos\alpha\cos\beta\right) = 22.63\,^{\circ}$. The last step is finding out how this affects the change in the direction at which Mišo observes the reflection. It turns out that it is exactly the angle γ.

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Problem FoL.22 ... Red hot nickel... cable? 4 points

A heater has a nickel heating filament with a temperature coefficient of resistance $\alpha = 0.0068\,\mathrm{K}^{-1}$ and with a resistance $R_0 = 80\,\Omega$ at the room temperature $T_0 = 293\,\mathrm{K}$. After turning on the heater, the filament becomes red-hot at the temperature $T = 1100\,\mathrm{K}$ thanks to a current $I = 2\,\mathrm{A}$ passing through it. What is the total surface of the filament (in m^2), assuming that the filament transfers heat only by radiation (acts as a black body)?

Jirka is already thinking about how to warm up in winter.

The power of the hot filament as a resistor is $P = RI^2$. This power will be the same as the radiant flux transmitted by the filament to its surroundings $\Phi = \sigma T^4 \cdot S$. However, we still do not know the resistance of the hot filament. This can be calculated using the formula $R = R_0(1 + \alpha\Delta t)$, where we substitute $T - T_0$ for Δt. We get the equation

$$\sigma T^4 \cdot S = R_0[1 + \alpha(T - T_0)]I^2 \,,$$

from which we simply express the required area

$$S = \frac{R_0[1 + \alpha(T - T_0)]I^2}{\sigma T^4} \,.$$

Now, all we have to do is substitute the numeric values

$$S = \frac{80\,\Omega[1 + 0.0068\,\mathrm{K}^{-1} \cdot (1100\,\mathrm{K} - 293\mathrm{K})] \cdot (2\,\mathrm{A})^2}{5.67 \cdot 10^{-8}\,\mathrm{W} \cdot \mathrm{m}^{-2} \cdot \mathrm{K}^{-4} \cdot (1100\,\mathrm{K})^4} = 0.025\,\mathrm{m}^2 \,.$$

The filament's surface area is $0.025\,\mathrm{m}^2$.

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Problem FoL.23 ... hanging here

5 points

Two springs with identical length $l = 5.0\,\mathrm{m}$ and stiffness $k = 1.0.10^5\,\mathrm{N{\cdot}m^{-1}}$ are stretched horizontally between two vertical walls (at the distance $2l$) in such a way that the springs are connected in the middle. Next, we hang a mass $m = 3.0\,\mathrm{kg}$ at the point where the springs are connected. What is the height h by which the point of connection drops? You may assume that $h \ll l$ and $g = 9.81\,\mathrm{m{\cdot}s^{-2}}$.

The tension is rising.

The length of each spring under the load is $l' = \sqrt{l^2 + h^2}$, so each spring is extended by

$$\Delta l = l' - l = \sqrt{l^2 + h^2} - l = l\left(\sqrt{1 + \frac{h^2}{l^2}} - 1\right) \approx l\left(1 + \frac{h^2}{2l^2} - 1\right) = \frac{h^2}{2l} \,.$$

The vertical component F_v of the force with which the springs are acting on the mass is

$$F_v = \frac{h}{\sqrt{l^2 + h^2}} k\Delta l = \frac{kh^3}{2l^2}\frac{1}{\sqrt{1 + \frac{h^2}{l^2}}} \approx \frac{kh^3}{2l^2}\left(1 - \frac{h^2}{2l^2}\right) \approx \frac{kh^3}{2l^2} \,.$$

The two springs together balance out with the force of gravity F_g acting on the mass, and hence

$$F_g = 2F_v \,,$$

$$mg = \frac{kh^3}{l^2} \,,$$

$$h = \sqrt[3]{\frac{mgl^2}{k}} \approx 0.19\,\mathrm{m} \,.$$

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Problem FoL.24 ... doggy 6 points

Suppose that we are on a hypothetical planet Caniculium orbiting the white dwarf Sirius B, which is the weaker component of the Sirius binary system. From the orbit of this planet, a hybrid eclipse of Sirius A can be observed (in a hybrid eclipse, the stars have the same angular diameter). Find out the proportion of landscape illumination (power incident on the surface of the planet) during the eclipse compared with the situation just before the eclipse starts. The effective surface temperatures of these stars are $T_A = 9940\,\mathrm{K}$ $T_B = 25\,000\mathrm{K}$.

Dodo thinks that the Moon is not shining by itself.

A hybrid eclipse occurs when both stars involved in the eclipse have the same angular diameters, so their radii R_A, R_B and distances from the planet r_A, r_B have to satisfy

$$\frac{R_A}{r_A} = \frac{R_B}{r_B}\,.$$

We can determine the total radiant flux of a star from the Stephan-Boltzmann law as

$$L = 4\pi R^2 \sigma T^4\,.$$

This power is radiated to a whole sphere with radius r, so the radiant flux per $1\,\mathrm{m}^2$ received by the surface of the planet is

$$F = \frac{L}{4\pi r^2} = \frac{R^2}{r^2}\sigma T^4\,.$$

We are interested in the flux ratio between the situations where only Sirius B illuminates the planet and where both stars illuminate the planet

$$w = \frac{F_B}{F_A + F_B} = \left(1 + \frac{F_A}{F_B}\right)^{-1} = \left(1 + \frac{\frac{R_A^2}{r_A^2}\sigma T_A^4}{\frac{R_B^2}{r_B^2}\sigma T_B^4}\right)^{-1} = \frac{1}{1 + \frac{T_A^4}{T_B^4}} \doteq 0.976\,.$$

During the eclipse, the landscape is illuminated by $w \doteq 0.976$ of the light before the eclipse.

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Problem FoL.25 ... ethanol emissions 3 points

We want to completely burn $V_\mathrm{e} = 1.00\,\mathrm{dm}^3$ *of liquid ethanol in an oxygen rich atmosphere, so that the main product of the combustion is carbon dioxide* (CO_2)*. All the released carbon dioxide should be contained in a vessel at the standard atmospheric pressure and temperature* $0\,^\circ\mathrm{C}$. *What is the required volume of such a vessel? (Assume that we can eliminate all other reaction products.)*

Karel was thinking about experiments.

Let's denote the density and molecular weight of ethanol by $\varrho = 789\,\mathrm{kg{\cdot}m^{-3}}$ and $M_\mathrm{m} = 46.07\,\mathrm{g{\cdot}mol^{-1}}$ respectively. Then, the molar amount of ethanol is

$$n_\mathrm{e} = \frac{V_\mathrm{e}\varrho}{M_\mathrm{m}}\,.$$

One molecule of ethanol contains two carbons, so the amount of carbon dioxide will be twice the amount of ethanol, $n_\mathrm{o} = 2n_\mathrm{e}$. The last thing we need to do is multiply the molar amount of CO_2 and the molar volume $V_\mathrm{m} = 22.41\,\mathrm{dm^3 \cdot mol^{-1}}$ of a gas at the given standard conditions and we get

$$V = n_\mathrm{o} V_\mathrm{m} = \frac{2V_\mathrm{e}\varrho V_\mathrm{m}}{M_\mathrm{m}} \doteq 0.768\,\mathrm{m^3}\,.$$

To contain all the carbon dioxide produced by burning $1\,\mathrm{l}$ of ethanol, we would need a vessel with volume $V \doteq 0.768\,\mathrm{m^3}$.

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Problem FoL.26 ... jump into the unknown 4 points

A very thin cylinder with a length $l = 32.7\,\mathrm{cm}$ is in a vertical position with its bottom base at the height $10\,l$ above a water surface. What depth below the water surface does the lowest point of the cylinder reach after we let it fall? The density of the liquid is three times higher than the density of the cylinder. Neglect any drag, friction and surface tension.

Jáchym fell into the water in Minecraft.

Let us introduce a vertical coordinate x pointing downwards, with the origin at the surface of the water. The cylinder is subject to a force of gravity $F_g = mg$ and a buoyant force

$$F_\mathrm{v} = \begin{cases} 0\,, & x \in (-nl,\, 0) \\ -xS\varrho g\,, & x \in (0,\, l) \\ -lS\varrho g, & x \in (l,\, \infty)\,, \end{cases}$$

where $n = 10$, S is the cross-sectional area of the cylinder and ϱ is the density of water. The buoyant force is negative because it is directed in the direction of decreasing x. Also, let's denote $k = 3$; then, the density of the cylinder is ϱ/k and $m = lS\varrho/k$ holds.

The bottom of the cylinder dives to an unknown depth h, where it stops. At that point, it has zero kinetic energy, so its potential energy is the same as at the beginning. Hence, the total work performed by gravity and buoyancy during the movement is zero. This can be expressed by the equation

$$\int_{-nl}^{h} (F_g + F_\mathrm{v})\;\mathrm{d}x = 0\,.$$

We divide this integral into three parts, because the buoyant force behaves differently at three different intervals, and we get

$$\int_{-nl}^{0} mg\,\mathrm{d}x + \int_{0}^{l} (mg - xS\varrho g)\;\mathrm{d}x + \int_{l}^{h} (mg - lS\varrho g)\;\mathrm{d}x = 0\,,$$

$$[mgx]_{-nl}^{0} + \left[mgx - \frac{1}{2}x^2 S\varrho g\right]_{0}^{l} + [mgx - lS\varrho gx]_{l}^{h} = 0\,.$$

From this, we can easily express

$$h = \frac{l}{2}\frac{2nm + lS\varrho}{lS\varrho - m}\,.$$

Now we just substitute for m and we can write the result

$$h = \frac{l}{2}\frac{2n + k}{k - 1} = \frac{23}{4}l \doteq 1.88\,\mathrm{m}\,.$$

The cylinder sinks to the depth 1.88 m. Thanks to all simplifications in the problem statement, the result does not depend on its area or the shape of the base.

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Problem FoL.27 ... rolling cone

5 points

On a horizontal surface, a cone with a base radius $r = 10.0\,\mathrm{cm}$ *and height* $H = 100.0\,\mathrm{cm}$ *made from a material with density* $\varrho = 1\,253\,\mathrm{kg{\cdot}m^{-3}}$ *is slipping with negligible rolling friction. The angular speed of rotation of the cone around the axis perpendicular to the surface and passing through its apex is* $\omega = 2.50\,\mathrm{rad{\cdot}s^{-1}}$. *Determine the magnitude of the force which the cone exerts on the surface.*

Dodo's head was spinning.

The center of gravity of the cone moves uniformly along a circle, so the net force acting on the cone must be directed from the center of gravity perpendicularly to the axis of rotation. If the distance of the center of gravity from the axis is s, we have $F_\mathrm{c} = m\omega^2 s$ for the centripetal force. The cone is subject to the force of gravity F_g and the reactive force from the pad F. The horizontal component of the force from the pad is thus $F_1 = -F_\mathrm{c}$ and its vertical component is $F_2 = -F_g = -mg$. Overall, its magnitude is

$$F = \sqrt{F_1^2 + F_2^2} = m\sqrt{\omega^4 s^2 + g^2} = \frac{\pi}{3}\varrho r^2 h\sqrt{\frac{9\omega^4 h^4}{16(h^2 + r^2)} + g^2}\,,$$

where we used the Pytagorean theorem and the fact that the center of gravity of the cone divides the line segment connecting the apex and the centre of the base in a 3:1 ratio.

When we evaluate the equation, we get $F \doteq 143\,\mathrm{N}$.

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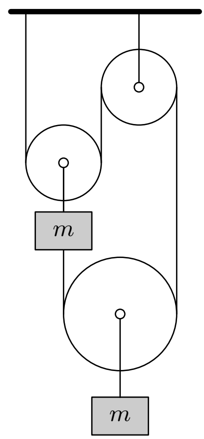
Problem FoL.28 ... pulleys

5 points



Find the acceleration of the lower pulley and its direction (the result should be negative if the pulley accelerates downwards or positive if it accelerates upwards). Neglect moments of inertia of the pulleys.

Matěj wanted to invent an infinite system of pulleys, but it didn't work.

Let us begin with the fact that the tensile force along the length of the rope is uniform and equal to F at each point. For the acceleration of the first pulley, we obtain

$$a_1 = \frac{F}{m} - g\,,$$

and for the lower one, we get

$$a_2 = \frac{2F}{m} - g\,.$$

Now we only have to realise that the rope is inelastic, so $a_1 = -2a_2$.

Solving these 3 equations, we get

$$a_1 = -2/5g\,,$$
$$a_2 = 1/5g = 1.96\,\mathrm{m{\cdot}s^{-2}}\,.$$

The acceleration of the lower pulley is $a_2 = 1.96\,\mathrm{m{\cdot}s^{-2}}$.

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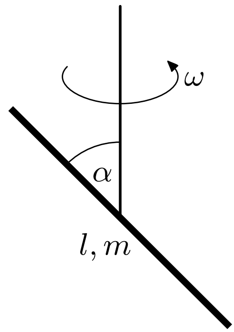
Problem FoL.29 ... small torque

5 points

A thin homogeneous rod is attached to an axis of rotation passing through the rod's centre. The angle between the rod and the axis of rotation is $\alpha = 45^\circ$. The rod has mass $m = 40\,\mathrm{g}$, length $l = 30\,\mathrm{cm}$ and it rotates with an angular velocity $\omega = 25\,\mathrm{rad{\cdot}s^{-1}}$. What is the magnitude of the torque by which the rod acts on its point of contact with the axis?

Jindra's problem with the inertia tensor didn't pass, so he invented this.



In the rotating reference frame connected with the rod, there is a centrifugal force acting on the rod, which pushes its mass from the axis of rotation. If the rod was perpendicular to the axis of rotation, the centrifugal forces acting on both ends of the rod would be collinear and their torque would be zero.

Let's denote the linear mass density of the rod by τ. The centrifugal force acting on an element of the rod is

$$\mathrm{d}F_\mathrm{o} = \mathrm{d}m\omega^2 r_x = \tau\omega^2 s \sin\alpha \mathrm{d}s\,.$$

The total torque can be calculated by integration of these infinitesimal forces' torques over the whole length of the rod.

$$\mathrm{d}M_\mathrm{o} = r_y \mathrm{d}F_\mathrm{o} = s\cos\alpha \mathrm{d}F_\mathrm{o} = \tau\omega^2 \sin\alpha\cos\alpha s^2 \mathrm{d}s$$

$$M_\mathrm{o} = \tau\omega^2\sin\alpha\cos\alpha\int_{-l/2}^{l/2} s^2\mathrm{d}s = \tau\omega^2\sin\alpha\cos\alpha\left[\frac{1}{3}s^3\right]_{-l/2}^{l/2}$$

$$M_\mathrm{o} = \frac{1}{24}m\omega^2 l^2\sin 2\alpha$$

$$M_\mathrm{o} = 0.09375\,\mathrm{N{\cdot}m}\,.$$

The torque acting on the point of connection between the rod and the axis is $0.09375\,\mathrm{N{\cdot}m}$.

The vector of the torque points perpendicularly to the plane containing the rod and the axis. In general, the vectors of angular momentum and angular velocity don't have to be parallel. In that case, the vector of angular momentum rotates around the axis. In order to prevent the rod from turning until it's perpendicular to the axis, the point of connection has to exert on it the same torque with the opposite direction. This torque causes the rotation of angular momentum in accordance with Newton's second law (for rotational motion) $M = \frac{\mathrm{d}L}{\mathrm{d}t}$.

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Problem FoL.30 ... asteroid with impact parameter 5 points

An asteroid very, very far from the Sun (we may assume it is infinitely far) has a speed $v = 3.9\,\mathrm{km\cdot s^{-1}}$*. If there was no gravitational influence on the asteroid, the smallest distance between the asteroid and the Sun would be* 1 au*. What will be the smallest distance between the asteroid and the Sun when the Sun's gravitational force is acting on the asteroid?*

Karel was thinking about alien spacecrafts.

The angular momentum of the asteroid at infinity can be expressed as $L_1 = mr_1v_1$, where m is the mass of the asteroid, $v_1 = v$ and r_1 is the smallest distance between the Sun and the asteroid in the case with a straight-line trajectory.

For a real hyperbolic trajectory, the angular momentum is conserved, and at the asteroid's perihelium, the angular momentum is $L_2 = mr_2v_2$, where r_2 is the smallest distance of the asteroid from the Sun, so we get

$$r_1v_1 = r_2v_2\,.$$

The other conserved quantity is the total mechanical energy. The gravitational potential energy is defined as

$$E_\mathrm{p} = -\frac{GMm}{r}\,,$$

where $M = 1.99\cdot 10^{30}\,\mathrm{kg}$ is the mass of the Sun and we set E_p to zero at infinity. From the energy conservation,

$$v_1^2r_2^2 + 2GMr_2 - r_1^2v_1^2 = 0\,,$$

and by substituting for v_1, we get

$$v_1^2r_2^2 + 2GMr_2 - r_1^2v_1^2 = 0\,,$$

which has two solutions

$$r_2 = \frac{-GM \pm \sqrt{G^2M^2 + r_1^2v_1^4}}{v_1^2}\,.$$

Because distance from the Sun is always positive, we choose the + sign and get the final result

$$r_2 = \frac{-GM + \sqrt{G^2M^2 + r_1^2v_1^4}}{v_1^2} \doteq 1.3\cdot 10^6\,\mathrm{km}\,.$$

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Problem FoL.31 ... defective spherical mirror 5 points

An astronomer wants to photograph a star using a Newtonian telescope. The primary mirror of the telescope is spherical (not parabolic) with a radius of curvature $R = 1.00\,\mathrm{m}$ *and transverse diameter* $d = 10.0\,\mathrm{cm}$*. The astronomer placed a CCD sensor in the focal plane at a distance* $R/2$ *from the mirror's vertex. What will be the radius of the star's image on the chip? Assume that the star is a point source of light.* *Jindra was thinking about imperfections of the world…*

The spherical mirror is influenced by the so-called spherical aberration - the rays reflected from the edge of the mirror intersect at a different point than the rays reflected closer to the optical axis. The resulting image will be blurred. However, the law of reflection still holds, so we can calculate this effect. A ray coming from the star at infinity, parallel to the optical axis and at a distance x from the axis, forms an angle $\alpha = \arcsin(x/R)$ with the normal to the mirror's surface at the point of its reflection. The angle between the reflected beam and the optical axis is 2α and the ray crosses the optical axis at a distance $s = R/(2\cos\alpha)$ from the centre of curvature. The distance between the point where the beams passing very close to the optical axis intersect and the point where the beams reflected from the edge of the mirror intersect is

$$\Delta s = \frac{R}{2}\left(\frac{1}{\cos\alpha_m} - 1\right) ,$$

where $\alpha_\mathrm{m} = \arcsin(d/(2R))$, numerically $\alpha_\mathrm{m} \doteq 2.87^\circ$, and $\Delta s \doteq 6.26\cdot 10^{-4}\,\mathrm{m}$. The reflected rays create a circle on the CCD chip and its radius is

$$\varrho = \Delta s \tan(2\alpha_m) .$$

The numerical result is $\varrho = 62.9\,\mathrm{\mu m}$.

A parabolic mirror would solve the spherical aberration. On the other hand, manufacturing a spherical surface is easier and cheaper. Therefore, only mirrors with a larger diameter are made parabolic.

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Problem FoL.32 ... quantum Kepler — 5 points

As you surely know, Kepler's third law is the relationship between the semi-major axis and the orbital period of celestial bodies $a^3/T^2 = \mathrm{const}$. This relationship can be also derived from the nature of the gravitational field. The electric field is also inversely proportional to the square of distance, so the situation is analogous to the gravitational field. Derive how Kepler's third law would look for the system of an electron and a proton if they behaved according to the classical laws of physics. What would be the ratio of the third power of the semi-major axis and the square of the orbital period?

Jindra wondered if hydrogen orbitals could be explained using Platonic bodies.

Let the distance between the proton and the electron be a. They're orbiting their common center of gravity and its distance from the electron is

$$r_\mathrm{e} = \frac{m_p}{m_p + m_e} a .$$

The electron and proton attract each other with a force

$$F_\mathrm{E} = \frac{1}{4\pi\varepsilon_0}\frac{e^2}{a^2} .$$

At the same time, the attractive electric force acts as the centripetal force curving the paths of the electron and proton.

$$\frac{1}{4\pi\varepsilon_0}\frac{e^2}{a^2} = m_\mathrm{e}\omega^2 r_\mathrm{e} = m_\mathrm{e}\frac{4\pi^2}{T^2}\frac{m_\mathrm{p}}{m_\mathrm{p}+m_\mathrm{e}} a$$

$$\frac{a^3}{T^2} = \frac{e^2(m_\mathrm{p}+m_\mathrm{e})}{16\pi^3\varepsilon_0 m_\mathrm{e} m_\mathrm{p}} \qquad (1)$$

Since we want to know the result with high precision, at least 6 significant digits, the constants we plug into the formula need to have at least 6 significant digits too.[1]

$$e = 1.602\,176\,634 \cdot 10^{-19}\,\mathrm{C}$$
$$m_\mathrm{e} = 9.109\,383\,7015(28) \cdot 10^{-31}\,\mathrm{kg}$$
$$m_\mathrm{p} = 1.672\,621\,923\,69(51) \cdot 10^{-27}\,\mathrm{kg}$$
$$\pi \doteq 3.141\,592\,654$$
$$\varepsilon_0 = 8.854\,187\,8128(13) \cdot 10^{-12}\,\mathrm{F{\cdot}m}$$

After plugging into the equation (1), we get

$$\frac{a^3}{T^2} = 6.418\,74\,\mathrm{m^3{\cdot}s^{-2}}\,.$$

For the system of an electron and a proton (a hydrogen atom), the ratio is $a^3/T^2 = 6.418\,74\,\mathrm{m^3{\cdot}s^{-2}}$. Just for comparison, for the system of the Earth and the Sun, it is $a^3/T^2 = 3.36 \cdot 10^{18}\,\mathrm{m^3{\cdot}s^{-2}}$.

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Problem FoL.33 ... tense tree

6 points

Jáchym cut down a homogeneous cylindrical tree with height $l = 30.0\,\mathrm{m}$. The trunk of the tree is still partially connected to the stump, so the top of the tree follows a circular trajectory. Find the distance from the stump to the point where the normal stress in the tree is zero at the moment when the angle between the tree and the ground is $\alpha = 47^\circ$. The connection between the trunk and the stump isn't causing any energy losses. *Jáchym fell(ed a tree).*

Let's denote the distance from the stump to the point in question by x. Then, the center of gravity of the top segment of the tree with length $(l - x)$ and mass m_x is located at the distance $r = (l + x)/2$ from the stump. There is a vertical force of gravity $F_g = m_x g$ acting on this segment and the projection of its vector on the tree is $F_{g_t} = F_g \sin\alpha$. There needs to be a centripetal force $F_\mathrm{d} = m_x\omega^2 r$ acting on the segment to make it move along a circular arc, where ω is the angular velocity of the tree's rotation. At the distance x, there will be zero normal stress only when both forces acting on the top segment of the tree in the radial direction are equal. In other words, $F_{g_n} = F_\mathrm{d}$ must hold.

We can easily work out the value of angular velocity because the rotational energy of the tree is equal to the decrease of its potential energy

$$\frac{1}{2}J\omega^2 = mg\left(l - l\sin\alpha\right)/2\,,$$
$$\omega^2 = \frac{3g\left(1 - \sin\alpha\right)}{l}\,,$$

[1] https://en.wikipedia.org/wiki/List_of_physical_constants

where we used the moment of inertia of a bar with respect to its endpoint

$$J = \frac{1}{3} m l^2 \,.$$

Now we just substitute the given values into the force balance equation and we get

$$x = \frac{5 \sin \alpha - 3}{3 \left(1 - \sin \alpha\right)} l \doteq 24.4\,\mathrm{m}\,.$$

It is worth noting that the final formula does not make good sense for too small or too large angles α, because the distance x is either negative or higher than l in those cases. In our case, the value of α was chosen reasonably and we get $x = 24.4\,\mathrm{m}$.

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Problem FoL.34 ... slim line

5 points

Jindra is returning from a space mission and is moving towards the Earth with the speed $v = 0.300\,c$*. Danka, whose rest mass is* $50.0\,\mathrm{kg}$*, is looking forward to meeting Jindra after a long time, so she set off in a rocket towards Jindra's spaceship. However, Danka has been on a diet. She doesn't want Jindra to be able to measure her relativistic mass being greater than* $60.0\,\mathrm{kg}$*. What maximal speed* u *with respect to the Earth can Danka fly? The answer is the ratio between* u *and* c*.*

For Jindra, travelling around Earth is not enough.

Danka's relativistic mass, measured by Jindra in his reference frame, must not exceed $60.0\,\mathrm{kg}$. Let us denote the relative velocity of Danka with respect to Jindra as w. Jindra, from his frame of reference, observes that Earth is approaching him with velocity of $v = 0.300\,\mathrm{c}$ and Danka approaching him at a higher speed w. The velocities u and v are added together relativisticly, so

$$w = \frac{v + u}{1 + \frac{uv}{c^2}} \,. \tag{2}$$

The relationship between rest mass and relativistic mass, as measured by Jindra, is

$$m = \frac{m_0}{\sqrt{1 - \left(\frac{w}{c}\right)^2}} \,.$$

If Jindra is supposed to observe that Danka's weight is less than $60.0\,\mathrm{kg}$, then the maximal velocity at which she can move towards him is $w = c\sqrt{1 - (m_0/m)^2} = 0.553\,\mathrm{c}$. We express velocity u from equation (2) as

$$u = \frac{w - v}{1 - \frac{wv}{c^2}} \doteq 0.303\,\mathrm{c}\,.$$

Therefore, Danka can travel towards Jindra at maximum speed $u = 0.303\,\mathrm{c}$.

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Problem FoL.35 ... overheated flywheel 4 points

By how many degrees Kelvin will a flywheel heat up by stopping? The initial frequency of rotation of the flywheel is $f = 120\,\mathrm{Hz}$. *Assume that half of its kinetic energy will be used for heating it up. The flywheel is a homogeneous cylinder with a radius* $r = 15\,\mathrm{cm}$ *and it is made of steel with a specific heat capacity* $c = 450\,\mathrm{J\cdot kg^{-1}\cdot K^{-1}}$. *Hint: we don't need to know any other quantities, only the value of* π. *Karel was discussing IYPT tasks.*

The kinetic energy of the flywheel is

$$E = \frac{1}{2} J\omega^2 \,.$$

We know that $\omega = 2\pi f$ and the moment of inertia of a cylinder is $J = mr^2/2$, therefore

$$E = \pi^2 m r^2 f^2 \,.$$

We can determine the change in temperature from the expression for heat $Q = mc\Delta t$, where Δt is the change in temperature. From the problem statement, we know that we should only consider half of the dissipated energy, so we can calculate

$$\frac{E}{2} = Q \quad \Rightarrow \quad mc\Delta t = \frac{1}{2}\pi^2 m r^2 f^2 \quad \Rightarrow \quad \Delta t = \frac{\pi^2 r^2 f^2}{2c} \doteq 3.6\,\mathrm{K}\,.$$

The flywheel will heat up by about 3.6 K when it completely stops.

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Problem FoL.36 ... looping 6 points

Imagine a biker who wants to ride around a thin vertical circular looping (a cylindrical surface such that its axis is horizontal) with a radius $R = 10.0\,\mathrm{m}$. *Therefore, the looping has no entry or exit and the biker starts from the lowest point with a constant angular acceleration. What is its minimum value such that the motorcycle can drive through the looping while always keeping in contact with the looping (without falling)? Use* $g = 9.81\,\mathrm{kg\cdot s^{-2}}$.

Matěj has driven through a looping by car many times.

Let's solve the problem from the perspective of the biker. There are two forces acting on him: the centrifugal force and gravitational force. These forces can be separated into components parallel to the direction of velocity (tangent) and components perpendicular to the velocity (normal). Our first condition is that the total normal force must always push the biker to the track, otherwise he would fall off the track. Next, instead of forces, we will work only with accelerations so that we don't have to consider the biker's weight.

Let the radius of the looping be R. In the rotating system associated with the biker, there is a centrifugal acceleration and and Euler's acceleration, which act only in the tangential direction. The Coriolis acceleration is equal to zero because the radial velocity is equal to zero. The centrifugal acceleration has always only the normal component $a_\mathrm{c} = \omega^2 R = \varepsilon^2 t^2 R$, where $\omega = \varepsilon t$ is the angular velocity of the biker and ε is the angular acceleration of his motorcycle (with respect to the center of the cylinder). The normal component of the force of gravity is $a_\mathrm{n} = g\cos\varphi$, where φ is the angle describing the biker's position (zero at the start and $\varphi = \pi$ at the highest point). We can write down the condition of the biker sticking to the track as

$$a_\mathrm{n} + a_\mathrm{c} \geq 0\,,$$
$$g\cos\varphi + \varepsilon^2 t^2 R \geq 0$$
$$g\cos\varphi + 2\varepsilon R\varphi \geq 0$$

where using substitution $\varphi = \frac{\varepsilon t^2}{2}$ we get

$$\varepsilon \geq -\frac{g}{2R}\frac{\cos\varphi}{\varphi}\,. \tag{3}$$

This condition must hold for all φ, so we are looking for φ_m such that the expression $\frac{\cos\varphi_\mathrm{m}}{\varphi_\mathrm{m}}$ has the minimal value. Using the first derivative of the expression above, we get

$$\frac{-\varphi\sin\varphi - \cos\varphi}{\varphi^2} = 0\,,$$
$$\varphi\tan\varphi = -1\,,$$
$$\varphi = -\cot\varphi\,.$$

This type of equations cannot be solved analytically, but we can use a calculator, since the result is not needed with a high number of significant digits. After a few iterations, we find out that the lowest positive value that satisfies the equation is approximately $\varphi_\mathrm{m} = 2.798\,3\ldots$ Surprisingly, the critical point is not required to lie at the top of the looping.

After substituting this value into the equation (3), we get the minimum possible acceleration

$$a = \left(0.01683\,\mathrm{m^{-1}}\right) g = 0.16506\,\mathrm{rad{\cdot}s^{-2}}\,.$$

With a smaller acceleration, the biker would fall off the track.

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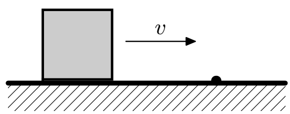
Problem FoL.37 ... bumps on the road — 6 points



A homogeneous cube with edge length $a = 1.0\cdot 10^{-2}$ m *is happily sliding with a velocity* v *on a horizontal plane without friction such that one face (hence two) is perpendicular to the direction of movement. Directly in front of it, there is a low stopper, and the cube collides with it perfectly inelastically; the point of impact is in the middle of its bottom front edge. Find the minimum velocity* v *such that the cube flips over the bump (stopper).* *Matej calculated this during a college party.*

Consider the conservation of angular momentum with respect to the axis which coincides with the edge that collided with the bump, since the cube started rotating around this axis as a result. Due to the fact that all points are moving with the velocity v before the collision and the direction of the velocity of the center of mass is at the distance $a/2$ from this axis, the angular momentum before the collision is

$$L = \frac{1}{2}mav\,,$$

where m is the mass of the cube. The moment of inertia of the cube with respect to the axis of rotation is, using Steiner's theorem, the sum of the moment of inertia of a cube around an axis passing through its center of mass and $m(a/\sqrt{2})^2$, because the center of mass is at the distance $a/\sqrt{2}$ from the axis,

$$J = \frac{1}{6}ma^2 + \frac{1}{2}ma^2 = \frac{2}{3}ma^2\,.$$

At the moment of collision, the block is turning around the bump with the angular velocity

$$\omega = \frac{L}{J}\,.$$

Its kinetic energy is

$$E = \frac{1}{2}J\omega^2 = \frac{L^2}{2J}\,.$$

This energy has to be high enough for it to go around the bump, what means to lift the center of gravity by

$$\frac{\sqrt{2}-1}{2}a\,,$$

so

$$E = \frac{\sqrt{2}-1}{2}amg\,.$$

After substituting this value into the previous formula for energy, we get

$$\frac{L^2}{2J} = \frac{\sqrt{2}-1}{2}amg\,,$$

$$\frac{3}{8}v^2 = \left(\sqrt{2}-1\right)ag\,,$$

$$v = \sqrt{\frac{8}{3}\left(\sqrt{2}-1\right)ag} = 0.3292\,\mathrm{m\cdot s^{-1}}\,.$$

This is the critical velocity at which the cube flips over.

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Problem FoL.38 ... speed merchant

4 points

Dan is driving on a straight road with a velocity $v_\mathrm{d} = 90\,\mathrm{km\cdot h^{-1}}$. In the distance $d = 110\,\mathrm{m}$ in front of him, he notices a cyclist riding with a velocity $v_\mathrm{c} = 18\,\mathrm{km\cdot h^{-1}}$. Because there are cars coming from the opposite direction, Dan can't drive around the cyclist, so he has to hit the brakes. His response time is $t_\mathrm{r} = 0.50\,\mathrm{s}$. What is the lowest allowed value of acceleration (deceleration due to the brakes) of the car such that Dan doesn't crush the cyclist? The car decelerates uniformly. *Jindra rode with Dan in a car once and had enough.*

The problem is easiest to solve if we use the reference frame connected with the cyclist. The velocity with which the car is approaching the cyclist is $v_\mathrm{r} = 72\,\mathrm{km\cdot h^{-1}} = 20\,\mathrm{m\cdot s^{-1}}$. Dano won't run over the cyclist if he brakes to zero (in the cyclist's reference frame) just behind him. Before Dano hits the brakes, the car travels a distance

$$d_1 = v_\mathrm{r} t_\mathrm{r} \,.$$

Numerically, $d_1 = 10\,\mathrm{m}$. In that moment, the distance between the car and the cyclist is $d_2 = 100\,\mathrm{m}$. We get the minimum acceleration from equations of motion with uniform acceleration

$$v_\mathrm{r}^2 = 2ad_2 \,,$$
$$a = \frac{v_\mathrm{r}^2}{2d_2} \,.$$

The minimum acceleration is $a = 2.0\,\mathrm{m\cdot s^{-2}}$.

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Problem FoL.39 ... slowly thrown rock — 6 points

Ida is one of the few asteroids in the main asteroid belt that has its own natural satelite. The moon's name is Dactyl. We haven't yet measured the shape of its orbit very precisely. Suppose that its distance from Ida in the pericenter is $r_\mathrm{p} = 100\,\mathrm{km}$ and in the apocenter is $r_\mathrm{a} = 200\,\mathrm{km}$. Dactyl's mass is $m_\mathrm{D} = 10^{10}\,\mathrm{kg}$, the mass of Ida is $m_\mathrm{I} = 5\cdot 10^{16}\,\mathrm{kg}$. Imagine an astronaut with a mass $m = 100\,\mathrm{kg}$ on the surface of Dactyl. In the pericenter, the astronaut jumps up in the direction of motion of Dactyl with a velocity $v = 5\,\mathrm{m\cdot s^{-1}}$. Determine the displacement of the apocenter distance of Dactyl caused by the jump (a positive number means that this distance increases). Neglect gravitational effects of the astronaut, i.e. treat it as a two-body problem.

Mirek gains inspiration at conference lectures.

We can estimate the new speed of Dactyl (in the pericenter, immediately after the jump) from the law of momentum conservation as

$$v_\mathrm{p}' = v_\mathrm{p} - \frac{m}{m_\mathrm{D}} v \,.$$

The next part of our solution is based on the laws of conservation of angular momentum and conservation of energy using the distances in the apocenter and pericenter. It is necessary to realize that if we subtract the weight of the astronaut from the weight of Dactyl, its orbit remains unchanged. Therefore, we may use conservation laws written with the mass of Dactyl

$$m_\mathrm{D} r_\mathrm{p} v_\mathrm{p} = m_\mathrm{D} r_\mathrm{a} v_\mathrm{a} \,,$$
$$-\frac{G m_\mathrm{D} m_\mathrm{I}}{r_\mathrm{p}} + \frac{1}{2} m_\mathrm{D} v_\mathrm{p}^2 = -\frac{G m_\mathrm{D} m_\mathrm{I}}{r_\mathrm{a}} + \frac{1}{2} m_\mathrm{D} v_\mathrm{a}^2$$

and after the astronaut jumps, we get the same pair of equations, but with new velocities v_p' instead of v_p, v_a' instead of v_a, and distances r_a' instead of r_a and $r_\mathrm{p}' = r_\mathrm{p}$. It is obvious that the new pericenter is the same as the original one, because the direction of velocity remains unchanged and the satellite moves perpendicularly to the line Dactyl-Ida only in case of being in the pericenter or apocenter (and it can not be in the new apocenter, since the change in the velocity was very small).

From the first set of laws (non-primed variables), we can express the velocity in the pericenter as

$$v_\mathrm{p} = \left(\frac{2Gm_\mathrm{I}}{r_\mathrm{p}\left(1+\frac{r_\mathrm{p}}{r_\mathrm{a}}\right)}\right)^{1/2} = 6.670\,\mathrm{m{\cdot}s^{-1}}\,,$$

therefore we also know the velocity v'_p. Furthermore, we can approximate

$$(v'_\mathrm{p})^2 \approx v_\mathrm{p}^2 - \frac{2m}{m_\mathrm{D}} v v_\mathrm{p}\,.$$

Similarly, we can express r'_a from the second set of laws (primed variables) as

$$r'_\mathrm{a} = -\frac{v'^2_\mathrm{p} r_\mathrm{p}}{v'^2_\mathrm{p} - \frac{2Gm_\mathrm{I}}{r_\mathrm{p}}} \approx r_\mathrm{a}\left(1 - \frac{4Gm_\mathrm{I} m v}{v_\mathrm{p}\left(v_\mathrm{p}^2 - \frac{2Gm_\mathrm{I}}{r_\mathrm{p}}\right) r_\mathrm{p} m_\mathrm{D}}\right)\,.$$

Now we can substitute v_p into the equation, but the expression for r'_a does not change much. Furthermore, we can see that v_p does not appear anywhere where subtraction of close numbers would occur, so we can use the numeric value calculated above and together with values given in the problem statement, we get

$$r'_\mathrm{a} - r_\mathrm{a} \doteq -4.45\cdot 10^{-8} r_\mathrm{a} \doteq -9.0\cdot 10^{-3}\,\mathrm{m}\,.$$

The apocenter of Dactyl would move by 0.9 cm closer to Ida. A relative change by 10^{-8} was expected, since the velocity of the astronaut's jump and the velocity of Dactyl in the pericenter are similar and the ratio of their masses is $m/m_\mathrm{D} = 10^{-8}$.

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Problem FoL.40 ... aesthetic balls 6 points

We have two small charged balls with the same mass hanging on strings (with negligible mass) from one point at the ceiling. Both balls have the same charge and they are in a homogeneous gravitational field. The angle between the strings is 10°. Then, we double the charge of one ball and increase the charge of the other ball k times. Find the value of k which is necessary if we want the angle between the strings to be 25°. *Karel heard about Danka's plans.*

Let's denote the initial charges, the masses of the balls and the lengths of the strings by q, m and l respectively. The initial slope of the string is $\alpha_0 = 5^\circ$. The system is stationary and therefore, the net force acting on each ball must be zero. To satisfy this, we only need the vector sum of the force of gravity and electrostatic force to point in the direction of the string. The forces are

$$F_g = mg\,,$$

$$F_\mathrm{e} = \frac{1}{4\pi\varepsilon}\frac{q^2}{4l^2\sin^2\alpha_0}\,,$$

and then we get

$$\frac{F_\mathrm{e}}{F_g} = \tan\alpha_0\,.$$

After simplification, we can write

$$\frac{1}{4\pi\varepsilon}\frac{q^2}{4l^2mg} = \tan\alpha_0 \sin^2\alpha_0\,.$$

Now we change the angle to $\alpha = 12.5^\circ$. All constants remain the same, the only difference is that q^2 changes to $2kq^2$. From the simplified equation, we can easily express

$$k = 4\pi\varepsilon\frac{4l^2mg}{2q^2}\tan\alpha\sin^2\alpha = \frac{\tan\alpha\sin^2\alpha}{2\tan\alpha_0\sin^2\alpha_0}\,,$$

which gives the result $k \doteq 7.81$.

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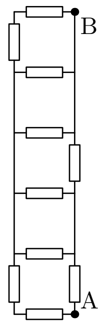
Problem FoL.41 ... disgusting resistive skyscraper — 7 points



Determine the resistance between the points A and B if each resistor has resistance $R = 3141.59\,\Omega$.

Matej was reminiscing about Physics Olympiad preparation.

The problem can be solved in a classical way according to the rules of resistor assembling[2] or according to Kirchhoff's laws. We would get the same result, but we won't explain these approaches because it's tedious and there are plenty of problems solved this way.

We can also do it with a non-traditional, but for this particular problem faster, approach. First, we will redraw the electrical circuit into an equivalent form in the figure. We deliberately drew resistors with different sizes because we can notice that each resistor corresponds to one square in the figure. Imagine that the whole square in this figure is made up of material with constant conductivity per area. If we apply electric voltage between its upper and lower side[3] then due to homogeneity, each horizontal line corresponds to a conductor in the original wiring of resistors, i.e. an equipotential. This implies that the height of each sub-square corresponds to the voltage on the corresponding resistor. By analogy, the width of each square[4] corresponds to the electric current passing through the corresponding resistor. However, these are all squares, each resistor has the same voltage to current ratio and therefore, all resistors have the same resistance. This fulfills the conditions given in the problem statement. The figure 2 then shows that the total resistance of the circuit is equal to the resistance of one resistor because the whole square also has the same aspect ratio.

Note: In this procedure, it is necessary to follow the rule that the aspect ratio of each individual rectangle (in our case, all of them are squares) is proportional to the resistance on the corresponding resistor and the rectangles must form one large rectangle without any holes.

[2] Using star-delta transformation is necessary.
[3] I.e. we connect wires with the given potential difference to both sides.
[4] In general, rectangles could also occur here.

Otherwise, we cannot use this procedure so simply. Then, the resulting resistance is derived from the aspect ratio of the total rectangle. This problem was created by exactly the opposite procedure to the solution mentioned above - we made a problem which is solvable this way - and therefore, it is easy to reverse this procedure, but generally, it doesn't have to work.

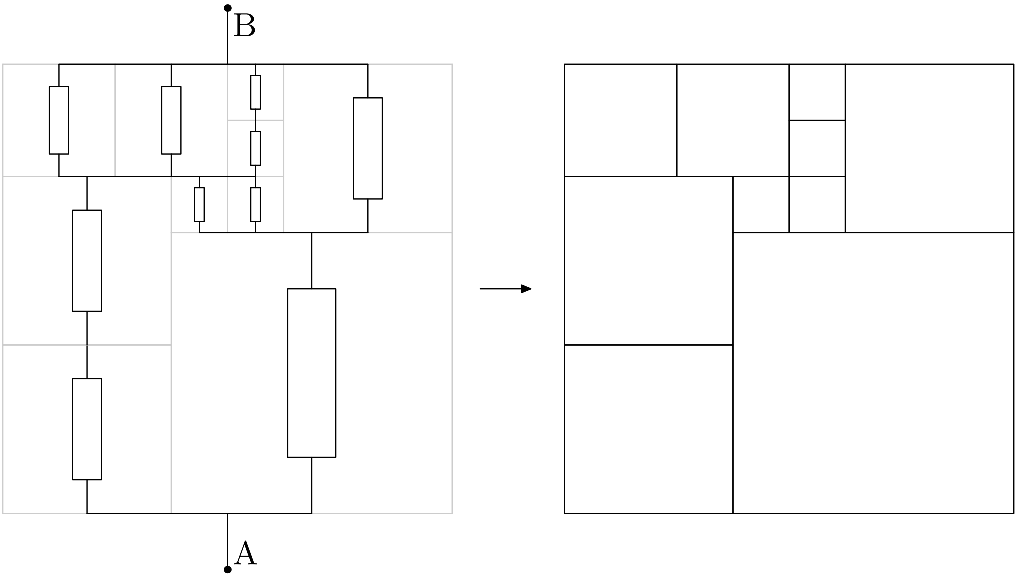


Fig. 2: Equivalent form of the circuit

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Problem FoL.42 ... manoeuvre

6 points

Imagine that we are operating a spacecraft on a circular orbit around a star and we wish to double the radius of our circular orbit. We choose to achieve this by a specific manoeuvre: we maintain a constant velocity – with a constant magnitude and a constant angle $\theta = 5^\circ$ between the velocity vector and the tangent line to the local circular orbit. How many orbits (not necessarily an integer) do we make before we reach the desired distance from the star?

Mirek came up with a problem which turned out to be too difficult for elementary school.

Let us solve this problem in polar coordinates. The velocity vector is given by $\boldsymbol{v} = v_0(\sin\theta, \cos\theta)$, where v_0 is the constant magnitude of the velocity – the value of this constant has obviously no effect on the result of this problem, so we set it to 1. The radial motion, as a time-dependent quantity, is described by a simple differential equation

$$\mathrm{d}r = v_0 \sin\theta \mathrm{d}t$$

with the unique solution $r(t) = r_0 + v_0 t \sin\theta$, where r_0 is the initial radial distance. The manoeuvre will take the time $\tau = r_0/\left(v_0 \sin\theta\right)$. The tangential motion is described by a slightly more complicated equation

$$\mathrm{d}\varphi = \frac{v_0 \cos\theta}{r(t)} \mathrm{d}t\,.$$

During the manoeuvre, the angular coordinate changes by

$$\Delta\varphi = \int_0^\tau \mathrm{d}\varphi = \int_0^\tau \frac{v_0 \cos\theta}{r_0 + v_0 t \sin\theta}\mathrm{d}t = \frac{1}{\tan\theta}\left[\ln\left(r_0 + v_0 t\sin\theta\right)\right]_0^\tau = \frac{\ln 2}{\tan\theta}\,.$$

For the given numerical value of θ, this amounts to $\Delta\varphi/(2\pi) = 1.26$ orbits.

Another approach to this problem is to recall that the requirement on constant θ defines a specific class of spirals – the trajectory is a spiral described by a formula

$$\ln r = \ln a \; + b\varphi$$

and called the logarithmic spiral. Here, a is an irrelevant constant and $b = \tan\theta$, as can be shown by differentiating the defining equation.

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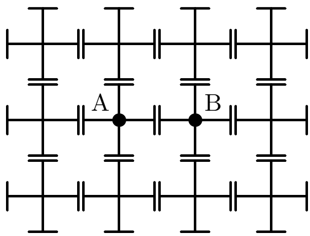
Problem FoL.43 ... capacitor grid 6 points

Consider a square lattice made from capacitors. Part of the lattice is shown in the figure. The capacitance of each capacitor is $C = 100.0\,\mathrm{\mu F}$. Find the total capacitance between two adjacent vertices.



Matěj would like to have an infinite grid at home.

It is convenient to solve the problem using a trick, as is the case with almost every problem where we are dealing with an infinite number of components and some kind of symmetry. We will use the principle of superposition for the electric field and separate the problem into two cases, which we will then combine.

In the first case, we add an external charge Q at the vertex A (while doing nothing with the vertex B) and this charge distributes uniformly between the four adjacent capacitors, because the lattice is symmetrical under rotation by $90\,^\circ$. On each of these capacitors, we now have a charge $Q/4$, that is, on the side connected with the vertex A, there is a charge $Q/4$, and on the opposite side, there is an induced charge $-Q/4$.

In the second case, we add a charge $-Q$ at the vertex B and do nothing with the vertex A. We get an analogous situation, with charges $-Q/4$ on the four corresponding capacitors.

Now let us combine these two situations. According to the superposition principle, electric charge, potential and some other electromagnetic quantities are additive, so if we sum up several solutions,[5] we'll get another valid solution of our problem. This implies that if we put a charge Q in the vertex A and a charge $-Q$ in the vertex B, then the charge on the capacitor between these two vertices will be $Q/2$ (i.e. sum of $Q/4$ and $-Q/4$, where we changed the second sign due to the fact that the opposite side of a capacitor induces the opposite charge). Using the definition of capacitance, we can calculate the voltage between A and B

$$U_{\mathrm{AB}} = \frac{Q}{2C}\,.$$

If we connect the whole grid to an electric circuit through the vertices A and B, it will behave as one large capacitor with capacity C_{tot}. Because we know the voltage between these two vertices when there is a charge Q on each of them, this capacity can be easily calculated,

$$C_{\mathrm{tot}} = \frac{Q}{U_{\mathrm{AB}}} = 2C = 200.0\,\mathrm{\mu F}\,.$$

After some further thinking, we would realize that if an infinite grid of capacitors shows a rotational symmetry, the capacity between two neighboring vertices depends only on the number of edges N adjacent to every vertex as $C_{\mathrm{tot}} = \frac{N}{2}C$. In our case, $N = 4$. If you're interested, there is a similar problem in our archive – problem EG from Fyziklani2019 – with a similar question, but for a triangular grid and with resistors instead of capacitors.

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Problem FoL.44 ... the fourth velocity 5 points

The particle velocity distribution in a gas is described by the Maxwell-Boltzmann distribution

$$f(v) = \sqrt{\frac{2}{\pi}}\left(\frac{M}{RT}\right)^{3/2} v^2 \mathrm{e}^{-\frac{Mv^2}{2RT}}\,.$$

There are several ways to determine an average velocity. We define the most probable velocity $v_{\mathrm{p}} = \sqrt{\frac{2RT}{M}}$, *the mean absolute velocity* $v_{\mathrm{s}} = \sqrt{\frac{8RT}{\pi M}}$ *and the mean quadratic velocity* $v_{\mathrm{k}} = \sqrt{\frac{3RT}{M}}$. *For the oxygen* O_2 *at temperature* $20\,\mathrm{^\circ C}$, *these velocities have values* $390\,\mathrm{m\cdot s^{-1}}$, $440\,\mathrm{m\cdot s^{-1}}$ *and* $478\,\mathrm{m\cdot s^{-1}}$ *respectively. However, we can also calculate the* n*-th moment of the Maxwell-Boltzmann distribution for any* n *and thus find some other average speed. What is the value of the mean cubic absolute velocity of oxygen under the same conditions?*

An oxygen molecule hit Jindra's eye.

[5] In this case, we define a solution as a stable distribution of charge.

The mean cubic velocity is determined by calculating the third moment of the Maxwell-Boltzmann distribution and taking the cube root of it.

$$v_{\mathrm{cu}} = \sqrt[3]{\int_0^\infty v^3 f(v)\mathrm{d}v}\,,$$

$$v_{\mathrm{cu}} = \sqrt[6]{\frac{2}{\pi}}\left(\frac{M}{RT}\right)^{1/2}\sqrt[3]{\int_0^\infty v^5 \mathrm{e}^{-\frac{Mv^2}{2RT}}\mathrm{d}v}\,,$$

$$\text{substituce:}\quad \tfrac{Mv^2}{2RT} = t \qquad \Rightarrow \qquad v^2 = \frac{2RTt}{M}\,,$$

$$\tfrac{Mv}{RT}\mathrm{d}v = \mathrm{d}t\,,$$

$$v\mathrm{d}v = \frac{RT}{M}\mathrm{d}t\,,$$

$$v_{\mathrm{cu}} = \sqrt[3]{4}\sqrt[6]{\frac{2}{\pi}}\left(\frac{RT}{M}\right)^{1/2}\sqrt[3]{\int_0^\infty t^2\mathrm{e}^{-t}\mathrm{d}t}\,.$$

You probably recognized the gamma function in the integral with the root at first. The gamma function is a generalization of the factorial to all real (and also complex) numbers except zero and negative integers, which it is not defined for. The gamma function is defined as $\Gamma(n) = \int_0^\infty x^{n-1}\mathrm{e}^{-x}\mathrm{d}x$. And what do the gamma function and factorials have in common? If n is some natural number, then $\Gamma(n) = (n-1)\,!$. The integral under the root is $\Gamma(3) = (3-1)\,! = 2$.

$$v_{\mathrm{cu}} = 2\sqrt[6]{\frac{2}{\pi}}\left(\frac{RT}{M}\right)^{1/2}$$

After plugging in the numerical values,

$$v_{\mathrm{cu}} = 512\,\mathrm{m{\cdot}s^{-1}}\,.$$

The mean cubic velocity of the oxygen molecules is $512\,\mathrm{m{\cdot}s^{-1}}$.

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Problem FoL.45 ... inside a spherical mirror

6 points

Imagine two concave spherical mirrors with a common axis, such that the mirrors are two parts of a sphere with a radius $r = 1\,\mathrm{m}$. When we place a point light source in the centre of the sphere, each mirror forms a reflected image of the source which coincides with the source itself. Find the distance of the source from the centre such that it coincides with its image formed after three reflections (but does not coincide with the image formed after one reflection).

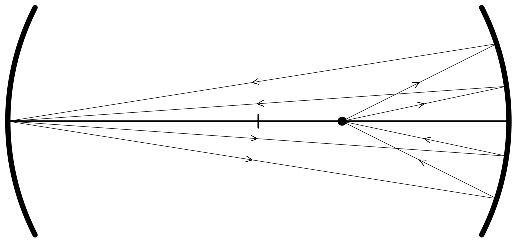
Matěj was inspired by a Vsauce video.

Let's imagine an optical axis passing through the centre of the sphere and through the source. We can replace the hollow sphere by two concave mirrors at distances r from the centre. Obviously, their focal distances are $\frac{r}{2}$.

We will use symmetry to solve this problem. After the first reflection, the first image should form at the distance r from the centre of the sphere and on the opposite side from the source (i.e. exactly in the centre of the second spherical mirror). Because of that, when beams are "sent" from this image and reflected for the second time, they return in the exactly opposite direction, which means that the first image is projected directly on itself (since it is located directly on the surface of the mirror). The third reflection sends the beams back to the source. Let's denote the distance of the light source from the centre of the sphere by x. We can write the mirror equation as

$$\frac{1}{r-x} + \frac{1}{2r} = \frac{2}{r}\,,$$

and its solution is $x = \frac{r}{3}$.



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Problem FoL.46 . . . rotary tree

5 points

Jachym cut down a tree that can be approximated by a thin homogeneous cylinder with length $l = 30.0\,\mathrm{m}$. The tree is still partly connected to the stump, so during its fall, the tip of the tree travels along a circular arc. There is a stone with negligible height lying on the ground at a distance $l/9$ away from the stump. Just before the tree hits the ground, it hits the rock in a perfectly inelastic manner. This breaks the connection between the tree and the stump and the tree continues to move as a free body. With what speed does the tip of the tree hit the ground? The connection to the stump or its breaking don't involve any losses in energy. *This year, Jáchym is going to get his Christmas tree from a forest.*

The moment of inertia of the tree with respect to a horizontal axis crossing the stump is

$$J_0 = \frac{1}{3}ml^2\,,$$

where m is mass of the tree. During free fall, its center of mass would drop from the height $h = 1/2$ directly above the ground, which corresponds to a potential energy drop by $E = mgh$. This energy would transform into rotational energy, so the angular velocity of the tree right before impacting the stone is

$$\omega_0 = \sqrt{\frac{2E}{J_0}} = \sqrt{\frac{3g}{l}}\,.$$

In the moment of impact, part of this rotational energy dissipates. However, the angular momentum with respect to the horizontal axis crossing the point of impact remains unchanged –

let's denote it by L. The point that hit the rock stops, therefore, the tree starts to rotate around it. Let's denote the moment of inertia of the tree with respect to the axis crossing the point of impact by J. Then, the angular velocity of the motion around the point of impact would be

$$\omega = \frac{L}{J}\,.$$

The velocity of the top of the tree can be easily calculated as $v = \omega\,(l - a)$, where $a = l/9$. Just before the impact, the velocity of a section of the tree at a distance x from the stub is equal to $\omega_0 x$. A section of the tree with length $\mathrm{d}x$ therefore has momentum $\mathrm{d}p = \lambda\omega_0 x\mathrm{d}x$, where $\lambda = m/l$ is the linear density of the tree. The angular momentum with respect to a point at a distance a from the stub is

$$L = \int_0^l (x-a)\,\mathrm{d}p = \int_0^l x\mathrm{d}p - a\int_0^l \mathrm{d}p = L_0 - a\,\lambda\int_0^l \omega_0 x\mathrm{d}x = L_0 - a\,\lambda\omega_0\frac{l^2}{2} = L_0 - \frac{1}{2}alm\omega_0\,,$$

where L_0 is the original angular momentum $J_0\omega_0$. The new moment of inertia can be calculated as

$$J = \int_0^l (x-a)^2\,\mathrm{d}m = \lambda\int_0^l (x-a)^2\,\mathrm{d}x = J_0 - alm + a^2m\,.$$

Now we only need to use all the equations together and we get

$$v = \omega\,(l-a) = \frac{L}{J}\,(l-a) = \frac{L_0 - \frac{1}{2}alm\omega_0}{J_0 - alm + a^2m}\,(l-a) = \frac{20}{19}\omega_0 l = \frac{20}{19}\sqrt{3gl} \doteq 31.3\,\mathrm{m{\cdot}s^{-1}}\,.$$

The top of the tree hits the ground with the velocity $31.3\,\mathrm{m{\cdot}s^{-1}}$.

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Problem FoL.47 ... neutron rack 7 points

Find the smallest distance from the surface of a neutron star at which a free-falling steel rod can exist, pointing towards the star, without being ripped in half. The star has mass $M = 1.80M_\odot$ *and radius* $R = 10.0\,\mathrm{km}$. *The ultimate tensile strength of the rod is* $\sigma = 800\,\mathrm{MPa}$, *its density is* $\varrho = 7\,900\,\mathrm{kg{\cdot}m^{-3}}$ *and its length is* $L = 1.00\,\mathrm{m}$. *Neglect deformation of the rod which occurs before fracturing and consider only Newtonian mechanics.*

Dodo was reading a sci-fi from 1967.

A famous similar problem deals with estimating the maximum length of a hanging cylinder (made of specified material) at which it doesn't fracture as a result of gravity. In our case, the rod is "hanging" in its center of gravity and stressed by tidal forces. Let's denote the height of the rod above the surface of the star by h. From the difference of gravitational forces affecting the rod at two points with distance l from each other, we get

$$a = \frac{GM}{(r+h)^2} - \frac{GM}{(r+h+l)^2} \approx \frac{GM}{(r+h)^3}2l\,,$$

which is our tidal acceleration.

The total tensile force acting on the material in the rod's center of gravity can be calculated by integration over layers with thickness $\mathrm{d}x$ and weight $\mathrm{d}m = \varrho\mathrm{d}x$, where S is the cross section of the rod. Let's describe the situation from the point of view of the center of mass of the rod, where we set $x = 0$. We consider the inhomogeneity of the field to be low enough that we can estimate the center of mass to be in the middle of the rod.

$$F = \int_0^{\frac{L}{2}} a\mathrm{d}m = \int_0^{\frac{L}{2}} \frac{GM\mathrm{d}m}{(r+h)^3} 2x = \int_0^{\frac{L}{2}} \frac{GM\varrho S}{(r+h)^3} 2x\mathrm{d}x\,,$$

$$F = \frac{GM\varrho S}{(r+h)^3} x^2 \Big|_0^{\frac{L}{2}} = \frac{GM\varrho S}{4(r+h)^3} L^2\,.$$

The force F acts away from the star on the farther half of the rod and symmetrically, it acts towards the star on the closer half of the rod. The tensile strength and force are connected by the equation $F = \sigma S$. After substituting for the force, we can calculate h and, assuming $r = R$, define the critical distance from the surface as

$$r = \sqrt[3]{\frac{GM\varrho L^2}{4\sigma}} - R \doteq 74\,\mathrm{km}\,.$$

We would also like to validate our approximations. In the first approximation, we introduce an uncertainity with magnitude 10^{-5}. Next, let's estimate the error in the position of the center of mass. We will calculate the "moment" of distribution of the force acting on the rod M_x around the middle of the rod

$$M_x = \int a(x)x\mathrm{d}m \approx \int_{\frac{-L}{2}}^{\frac{L}{2}} \left(F(0) + \frac{GM}{(r+h)^3} 2x\right) \varrho Sx\mathrm{d}x$$

$$M_x = 0 + \frac{2GM}{3(r+h)^3} \varrho S x^3 \Big|_{-\frac{L}{2}}^{\frac{L}{2}} = \frac{GM}{6(r+h)^3} \varrho S L^3\,.$$

In the center of mass, the value above should be zero. We can compare it with a rod in a homogeneous gravitational field a with respect to a point displaced from the middle of the rod by $s \ll L$

$$M_x(s) = \int_{\frac{-L}{2}-s}^{\frac{L}{2}-s} a\ \varrho Sx\mathrm{d}x = a\varrho S\left(\left(\frac{-L}{2} - s\right)^2 - \left(\frac{L}{2} - s\right)^2\right) \approx a\ \varrho S 2Ls\,.$$

Comparing with the tidal acceleration formula mentioned above $a = \frac{GM}{(r+h)^2}$, we have

$$s = \frac{L^2}{12(r+h)} \approx 1 \cdot 10^{-6}\,\mathrm{m}\,.$$

We can see that the approximations used in the solution are correct. The approximations given in the problem statement are worse, because the distance from the center of the neutron star equals approx. 17 Schwarzschild radii of the star (it would be necessary to consider the consequences of relativistic effects), and steel can stretch by tens of percents before tearing.

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Problem FoL.48 ... sunbathing session

6 points

The average solar irradiance of the Earth (power incident on the Earth's surface) oscillates with a period of about 11 years. A big part of the variance constitutes of changes in the UV part of the EM spectrum. Assume, for the purposes of this problem, that the Sun in the solar maximum is a blackbody with a temperature $T = 5.800\,\mathrm{K}$. *In the solar minimum, the spectral characteristic is the same, except for a region of wavelengths between* $10\,\mathrm{nm}$ *and* $300\,\mathrm{nm}$, *which is missing. Determine the total drop in power emitted by the Sun between the maximum and minimum, in percent of the value in the maximum.*

Mirek gains inspiration at conference lectures.

Black body radiation is described by Planck's law, which gives the specific spectral intensity of radiation as

$$I_\nu = \frac{2h\nu^3}{c^2}\frac{1}{\mathrm{e}^{\frac{h\nu}{k_\mathrm{B}T}} - 1}\,.$$

Since we only want to find the ratio of incident powers with a different range of frequencies, we only need to integrate over frequency (i.e. we do not calculate the exact power, but power per incident area). We can neglect the constant before the integral and substitute frequency by energy in electronvolts. Hence, the desired power drop is

$$\delta P = \frac{\int_{x_0}^{x_1} \frac{x^3 \mathrm{d}x}{\mathrm{e}^x - 1}}{\int_0^\infty \frac{x^3 \mathrm{d}x}{\mathrm{e}^x - 1}}\,,$$

where the upper and lower limits of the variable x in the integral are obtained by converting from wavelengths using the formula

$$x = \frac{hc}{\lambda k_\mathrm{B} T}\,.$$

In the denominator of the expression for δP, we can identify the definition of the ζ-function, so the denominator equals $\Gamma(4)\zeta(4) = \pi^4/15$. A more detailed solution is

$$\int_0^\infty \frac{x^3 \mathrm{d}x}{\mathrm{e}^x - 1} = \int_0^\infty x^3 \mathrm{e}^{-x} \sum_0^\infty \mathrm{e}^{-nx} \mathrm{d}x = \sum_0^\infty (n+1)^{-4} \int_0^\infty ((n+1)\,x)^3\, \mathrm{e}^{-(n+1)x} \mathrm{d}\,(n+1)\,x =$$

$$= \sum_1^\infty n^{-4}\Gamma(4) = \Gamma(4)\zeta(4)\,.$$

The numerator can be calculated in approximate form as

$$\int_{x_0}^{x_1} \frac{x^3 \mathrm{d}x}{\mathrm{e}^x - 1} \approx \int_{x_0}^\infty \frac{x^3 \mathrm{d}x}{\mathrm{e}^x} = \mathrm{e}^{-x_0}\left(x_0^3 + 3x_0^2 + 6x_0 + 6\right)\,,$$

since $x_0 \doteq 8.269 \gg 1$ and $x_1 \gg x_0$. Thus, we get the result $\delta P \doteq 3.3\,\%$.

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Problem FoL.49 ... diabolical spring 8 points

Let's consider a straight elastic rubber rod (i.e. non-circular rubber band) with rest length $l = 9.57\,\mathrm{cm}$, spring constant $k = 11.08\,\mathrm{N{\cdot}m^{-1}}$, mass $m = 24.53\,\mathrm{g}$ and cross-sectional area $S = 5.70\,\mathrm{cm^2}$. One end of the rod is connected with a pad underneath it (not vertically glued to the pad, i.e. the only constraint is that this end cannot move away from the pad). Now, we fill the space around the rubber rod with water; the density of water is $\varrho = 998\,\mathrm{kg{\cdot}m^{-3}}$. What will the new equilibrium length of the rod be? Assume that the cross-sectional area of the rod always remains constant.

You thought the last year's evil spring problem was hard? I will show you hard... – Jáchym

The density of the spring is $m/lS \doteq 450\,\mathrm{kg{\cdot}m^{-3}} < \varrho$, so in the equilibrium state, the spring will vertically float right above the pad.

For the initial state (without influence of any external forces), we introduce an x-coordinate with the origin at the lower end of the rod; for the final state, we use a similar y-coordinate. Now we are looking for a function $y(x)$ which maps a part of the original rod at a coordinate x to its distance from the pad y after immersion into water. The solution of this problem will be the value $y(l)$.

A small part of the rod at a coordinate x (at a height $y(x)$) is pulled upward by a force equal to the buoyant force acting on the segment of the rod above this part. Symbolically written, this force is

$$F_\mathrm{v}(x) = S\varrho g\,(y(l) - y(x))\ .$$

However, this segment of the rod is also influenced by gravity. Unlike buoyancy, the force of gravity depends on the original length of the segment, instead of the volume, which is directly proportional to its new equilibrium length. Obviously,

$$F_g(x) = mg\frac{l-x}{l}\ .$$

The last force acting on this segment of the rod (downwards) is the elastic force. If a very small part of the spring right underneath this segment has an original length Δx, we can model it by a spring with a spring constant[6]

$$k_{\Delta x} = k\frac{l}{\Delta x}\ .$$

Its current length is $\Delta y \approx y(x+\Delta x) - y(x)$, which corresponds to the force

$$F_\mathrm{p}(x) = k_{\Delta x}\,(\Delta y - \Delta x) = kl\left(y'(x) - 1\right)$$

in the limit $\Delta x \to 0$. The net force acting on any segment of the spring must be zero, which means that we need to solve a simple differential equation

$$\begin{aligned} F_\mathrm{v} - F_g &= F_\mathrm{p}\,,\\ S\varrho g\,(y(l) - y) - mg\frac{l-x}{l} &= kl\left(y' - 1\right)\,,\\ y' + Ay &= \frac{B}{l}x + Ay(l) - B + 1\,, \end{aligned}$$

[6]This can be proved by splitting a whole spring stretched by a force F into two smaller springs, which also have to exert forces equal to F on each other.

which we simplified by introducing substitutions

$$A = \frac{S\varrho g}{kl}\,,$$
$$B = \frac{mg}{kl}\,.$$

The homogeneous solution is

$$y_\mathrm{H} = C\mathrm{e}^{-Ax}\,,$$

where C is some constant. Now we are looking for a unique particular solution in the form of a polynomial with degree one, or $y_\mathrm{P} = ax + b$. By substituting y for y_P, we get

$$a \ + Aax + Ab = \frac{B}{l}x + Ay(l) - B + 1\,,$$
$$a \ = \frac{B}{Al} = \frac{m}{S\varrho l}\,,$$
$$b = y(l) + \frac{1}{A}\left(-B + 1 - \frac{B}{Al}\right)\,.$$

The result is

$$y = y_\mathrm{H} + y_\mathrm{P} = C\mathrm{e}^{-Ax} + ax + b\,,$$

where we can determine the constant C from the condition $y(0) = 0$. It gives $C = -b$, so

$$y = -b\mathrm{e}^{-Ax} + ax + b\,.$$

In addition, substituting $x = l$ must give $y(l)$. This leads to the formula

$$y(l) = \frac{1}{A}\left(B + \left(1 - \frac{B}{Al}\right)\left(\mathrm{e}^{Al} - 1\right)\right) = \frac{m}{S\varrho} + \frac{k}{S\varrho g}\left(l - \frac{m}{S\varrho}\right)\left(\mathrm{e}^{\frac{S\varrho g}{k}} - 1\right) \doteq 11.15\,\mathrm{cm}\,,$$

and that's the final result of this problem.

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Problem FoL.50 ... the charge and the pentagram 9 points

Jindra has had enough with the free charges so he decided to try some magic. He took a regular pentagon with side length 1 *and cut out a pentagram. Then he caught a free point charge* $Q = 1.00 \cdot 10^{-6}\,\mathrm{C}$ *and placed it directly above the centre of the pentagram at a distance* $\frac{1}{2}\sqrt{\frac{5-\sqrt{5}}{10}}$. *What is the total electric flux through the pentagram?* *Jindra still doesn't give up...*

First, let us describe the geometry of a pentagon (and a pentagram). The inner angle at each vertex of a regular pentagon is $\frac{3\pi}{5}$. The central angle between adjacent vertices (each angle vertex-centre-adjacent vertex) is $\frac{2\pi}{5}$. The angle at each outer vertex of a pentagram is $\frac{\pi}{5}$. The inner vertices of a pentagram are also the vertices of a smaller pentagon. This pentagon will be important later, let's call it simply "smaller pentagon". The centre of the smaller pentagon is identical with the centre of the larger pentagon.

The side length of the larger pentagon is 1, so the radius of its circumscribed circle is

$$R_\mathrm{O} = \frac{1}{2\sin\frac{\pi}{5}}\,.$$

The circle inscribed in the smaller pentagon has a radius

$$\varrho = R_\mathrm{O}\cos\frac{2\pi}{5} = \frac{\cos\frac{2\pi}{5}}{2\sin\frac{\pi}{5}}\,. \tag{4}$$

Since we know that[7] $\cos\frac{2\pi}{5} = \frac{\sqrt{5}-1}{4}$ and $\sin\frac{\pi}{5} = \sqrt{\frac{5-\sqrt{5}}{8}}$, we can plug them into the equation (4).

$$\varrho = \frac{\frac{\sqrt{5}-1}{4}}{2\sqrt{\frac{5-\sqrt{5}}{8}}} = \frac{\sqrt{5}-1}{8}\cdot\sqrt{\frac{8}{\sqrt{5}(\sqrt{5}-1)}} = \frac{1}{2}\sqrt{\frac{\sqrt{5}-1}{2\sqrt{5}}}$$

$$\varrho = \frac{1}{2}\sqrt{\frac{5-\sqrt{5}}{10}} \tag{5}$$

Bingo! The distance z between the charge and the centre of the pentagram is the same as the radius of the circle inscribed in the smaller pentagon ϱ.

Now let's look at the directions of the vectors of electric field on the surface of the pentagram. The distance between the charge and the centre of the pentagram is $z = \frac{1}{2}\sqrt{\frac{5-\sqrt{5}}{10}}$. Consider a point on the pentagram's surface with a distance r from the centre. From the Pythagorean theorem, we obtain $R = \sqrt{z^2+r^2}$ for the distance of this point from the charge. The magnitude of the electric field at this point is

$$E = \frac{1}{4\pi\varepsilon_0}\frac{Q}{z^2+r^2}\,.$$

The magnitude of the component perpendicular to the pentagram is

$$E_\perp = \frac{1}{4\pi\varepsilon_0}\frac{Q}{z^2+r^2}\cdot\frac{z}{\sqrt{z^2+r^2}}$$

$$E_\perp = \frac{1}{4\pi\varepsilon_0}\frac{Qz}{(z^2+r^2)^{\frac{3}{2}}}\,. \tag{6}$$

The electric flux through the pentagram can be obtained by solving the integral $\int_S E_\perp \mathrm{d}S$ over the surface area of the pentagram. Let's divide the surface into two parts to make the calculation

[7] Really, try to plug the values into a calculator. One possible way to derive these expressions for goniometric functions is to use the formulae for $\sin 2x$, $\sin 3x$ and $\sin 4x$ obtained from DeMoivre's theorem to arrive at

$$\sin\frac{2\pi}{5} + \sin\frac{4\pi}{5} + \sin\frac{6\pi}{5} + \sin\frac{8\pi}{5} = 0\,.$$

This equality simply describes the fact that if we walk around a pentagon, we will return to the original point. The individual terms can be interpreted as y-coordinates of the vectors describing the sides of the pentagon. After a few simplifications, we obtain

$$2\sin\frac{2\pi}{5}\cos\frac{2\pi}{5}(4\cos^2\frac{2\pi}{5} + 2\cos\frac{2\pi}{5} - 1) = 0\,.$$

The value of $\cos\frac{2\pi}{5}$ can be obtained by solving a quadratic equation.

easier. Section A is the area inside the circumscribed circle of the inner pentagon. There, we can calculate the electric flux easily. Section B is the part of the pentagon outside this circle. The radius of the circumscribed circle of the smaller pentagon is $r_\mathrm{O} = \varrho/\cos\frac{\pi}{5} = 4\varrho/(\sqrt{5}+1)$.

The electric flux through the section A is the same as the flux through the spherical cap above it.[8] The full sphere would have a radius $R_\mathrm{K} = \sqrt{r_\mathrm{O}^2 + z^2}$. The height of the spherical cap is $h = R_\mathrm{K} - z$. The electric flux is then

$$\varphi_1 = \frac{Q}{\varepsilon_0}\frac{S_\mathrm{vrchlík}}{S_\mathrm{koule}} = \frac{Q}{\varepsilon_0}\frac{\sqrt{r_\mathrm{O}^2+z^2} - z}{2\sqrt{r_\mathrm{O}^2+z^2}}\,,$$

and we plug $z = \varrho$ and $r_\mathrm{O} = 4\varrho/(\sqrt{5}+1)$ into this to get

$$\varphi_1 = \frac{Q}{2\varepsilon_0} - \frac{Q}{2\varepsilon_0}\frac{1}{\sqrt{\left(\frac{4}{\sqrt{5}+1}\right)^2 + 1}}$$

$$\varphi_1 = \frac{Q}{2\varepsilon_0} - \frac{Q}{2\varepsilon_0}\frac{\sqrt{5}+1}{\sqrt{22+2\sqrt{5}}}\,. \tag{7}$$

The flux through the section B can be obtained by integrating over the circular arcs corresponding to each outer vertex of the pentagram. If the radius of a circular arc is r, its length is

$$l_1 = 2r\left(\frac{2\pi}{5} - \arccos\frac{\varrho}{r}\right).$$

There are five vertices, so the total length is

$$l = r\left(4\pi - 10\arccos\frac{\varrho}{r}\right).$$

The length l lies between $\frac{2\pi\varrho}{\cos\frac{\pi}{5}}$ for $r = \frac{\varrho}{\cos\frac{\pi}{5}}$ and $l = 0$ for $r = \frac{\varrho}{\cos\frac{2\pi}{5}}$. The flux through the section B can be calculated as the integral

$$\varphi_2 = \int_{\frac{\varrho}{\cos\frac{\pi}{5}}}^{\frac{\varrho}{\cos\frac{2\pi}{5}}} E_\perp l \,\mathrm{d}r$$

$$\varphi_2 = \frac{Qz}{4\pi\varepsilon_0}\int_{\frac{\varrho}{\cos\frac{\pi}{5}}}^{\frac{\varrho}{\cos\frac{2\pi}{5}}} \frac{r(4\pi - 10\arccos\frac{\varrho}{r})}{(z^2+r^2)^{\frac{3}{2}}}\mathrm{d}r$$

$$\varphi_2 = \frac{Qz}{\varepsilon_0}\int_{\frac{\varrho}{\cos\frac{\pi}{5}}}^{\frac{\varrho}{\cos\frac{2\pi}{5}}} \frac{r}{(z^2+r^2)^{\frac{3}{2}}}\mathrm{d}r - \frac{5Qz}{2\pi\varepsilon_0}\int_{\frac{\varrho}{\cos\frac{\pi}{5}}}^{\frac{\varrho}{\cos\frac{2\pi}{5}}} \frac{r\arccos\frac{\varrho}{r}}{(z^2+r^2)^{\frac{3}{2}}}\mathrm{d}r\,.$$

[8] The integrals $\varphi_1 = \int_0^\varrho E_\perp 2\pi r\mathrm{d}r = \frac{Qz}{2\varepsilon_0}\int_0^\varrho \frac{r}{(z^2+r^2)^{3/2}}\mathrm{d}r$ are trivial and left as an exercise to the reader. The results will be the same (which is to be expected).

We calculate each integral separately.

$$\int \frac{r}{(z^2+r^2)^{\frac{3}{2}}}\mathrm{d}r = -\int \mathrm{d}t = -t + C = -\frac{1}{\sqrt{z^2+r^2}} + C$$

$$\text{substituce:} \quad \frac{1}{\sqrt{z^2+r^2}} = t$$

$$\frac{r\mathrm{d}r}{(z^2+r^2)^{\frac{3}{2}}} = -\mathrm{d}t$$

$$\frac{Qz}{\varepsilon_0}\int \frac{r}{(z^2+r^2)^{\frac{3}{2}}}\mathrm{d}r = -\frac{Qz}{\varepsilon_0}\frac{1}{\sqrt{z^2+r^2}} + C \tag{8}$$

The other integral is

$$\int \frac{r\arccos\frac{\varrho}{r}}{(z^2+r^2)^{\frac{3}{2}}}\mathrm{d}r = -\frac{\arccos\frac{\varrho}{r}}{\sqrt{z^2+r^2}} + \int \frac{\varrho}{r\sqrt{z^2+r^2}\sqrt{r^2-\varrho^2}}\mathrm{d}r$$

$$\text{per partes:} \quad u = -\frac{1}{\sqrt{z^2+r^2}} \qquad u' = \frac{r}{(z^2+r^2)^{\frac{3}{2}}}$$

$$v = \arccos\frac{\varrho}{r} \qquad v' = \frac{-1}{\sqrt{1-\left(\frac{\varrho}{r}\right)^2}}\frac{-\varrho}{r^2} = \frac{\varrho}{r\sqrt{r^2-\varrho^2}}$$

The resulting integral cannot be solved in terms of elementary functions. However, we are lucky. We know that $z = \varrho$. After we plug it in, the integral becomes solvable. Magic!

$$\int \frac{\varrho}{r\sqrt{r^4-\varrho^4}}\mathrm{d}r = \frac{1}{\varrho^2}\int \frac{\mathrm{d}r}{\frac{r}{\varrho}\sqrt{\left(\frac{r}{\varrho}\right)^4-1}} = \frac{1}{2\varrho}\int \frac{2\frac{r}{\varrho^2}\mathrm{d}r}{\left(\frac{r}{\varrho}\right)^2\sqrt{\left(\frac{r}{\varrho}\right)^4-1}} = \frac{1}{2\varrho}\int \frac{\sinh t\mathrm{d}t}{\cosh t\sqrt{\cosh^2 t-1}} =$$

$$\text{substituce:} \quad \left(\frac{r}{\varrho}\right)^2 = \cosh t$$

$$\frac{2r}{\varrho^2}\mathrm{d}r = \sinh t\mathrm{d}t$$

$$= \frac{1}{2\varrho}\int \frac{\mathrm{d}t}{\cosh t} = \frac{1}{2\varrho}\int \sqrt{1-\tanh^2 t}\mathrm{d}t = \frac{1}{2\varrho}\int \frac{1-\tanh^2 t}{\sqrt{1-\tanh^2 t}}\mathrm{d}t = \frac{1}{2\varrho}\int \frac{\mathrm{d}y}{\sqrt{1-y^2}} =$$

$$\text{substituce:} \quad \tanh t = y$$

$$\frac{1}{\cosh^2 t}\mathrm{d}t = \mathrm{d}y$$

$$(1-\tanh^2 t)\mathrm{d}t = \mathrm{d}y$$

$$= \frac{1}{2\varrho}\arcsin y + C = \frac{1}{2\varrho}\arcsin\tanh t + C = \frac{1}{2\varrho}\arcsin\frac{\sqrt{\cosh^2 t-1}}{\cosh t} + C = \frac{1}{2\varrho}\arcsin\frac{\sqrt{\left(\frac{r}{\varrho}\right)^4-1}}{\left(\frac{r}{\varrho}\right)^2} + C$$

Thus we can write

$$\int \frac{\varrho}{r\sqrt{r^4-\varrho^4}}\mathrm{d}r = \frac{1}{2\varrho}\arcsin\frac{\sqrt{r^4-\varrho^4}}{r^2} + C$$

and plug this into the original integral

$$\int \frac{r \arccos \frac{\varrho}{r}}{(\varrho^2+r^2)^{\frac{3}{2}}} \mathrm{d}r = -\frac{\arccos \frac{\varrho}{r}}{\sqrt{\varrho^2+r^2}} + \frac{1}{2\varrho} \arcsin \frac{\sqrt{r^4-\varrho^4}}{r^2} + C\,. \tag{9}$$

Now we can combine the equations (8) and (9) and calculate the electric flux through the tips of the pentagram.

$$\varphi_2 = \frac{Q\varrho}{\varepsilon_0}\left[-\frac{1}{\sqrt{\varrho^2+r^2}} + \frac{5}{2\pi}\frac{\arccos\frac{\varrho}{r}}{\sqrt{\varrho^2+r^2}} - \frac{5}{2\pi}\frac{1}{2\varrho}\arcsin\frac{\sqrt{r^4-\varrho^4}}{r^2}\right]_{\frac{\varrho}{\cos\frac{\pi}{5}}}^{\frac{\varrho}{\cos\frac{2\pi}{5}}}$$

We can factor ϱ out of the square roots in the denominators and cancel them out. After further manipulation, we obtain

$$\varphi_2 = \frac{Q}{\varepsilon_0}\left(\frac{1}{2\sqrt{1+\frac{1}{\cos^2\frac{\pi}{5}}}} - \frac{5}{4\pi}\left(\arcsin\sqrt{1-\cos^4\frac{2\pi}{5}} - \arcsin\sqrt{1-\cos^4\frac{\pi}{5}}\right)\right).$$

Into this, we can plug the values of goniometric functions $\cos\frac{2\pi}{5} = \frac{\sqrt{5}-1}{4}$ and $\cos\frac{\pi}{5} = \frac{\sqrt{5}+1}{4}$.

$$\varphi_2 = \frac{Q}{2\varepsilon_0}\left(\frac{\sqrt{5}+1}{\sqrt{22+2\sqrt{5}}} - \frac{5}{2\pi}\left(\arcsin\sqrt{\frac{25+3\sqrt{5}}{32}} - \arcsin\sqrt{\frac{25-3\sqrt{5}}{32}}\right)\right)$$

The total electric flux is the sum of φ_1 and φ_2

$$\begin{aligned}
\varphi &= \frac{Q}{2\varepsilon_0} - \frac{Q}{2\varepsilon_0}\frac{\sqrt{5}+1}{\sqrt{22+2\sqrt{5}}} + \\
&+ \frac{Q}{2\varepsilon_0}\left(\frac{\sqrt{5}+1}{\sqrt{22+2\sqrt{5}}} - \frac{5}{2\pi}\left(\arcsin\sqrt{\frac{25+3\sqrt{5}}{32}} - \arcsin\sqrt{\frac{25-3\sqrt{5}}{32}}\right)\right) \\
\varphi &= \frac{Q}{2\varepsilon_0}\left(1 - \frac{5}{2\pi}\left(\arcsin\sqrt{\frac{25+3\sqrt{5}}{32}} - \arcsin\sqrt{\frac{25-3\sqrt{5}}{32}}\right)\right) \\
\varphi &= \frac{Q}{2\varepsilon_0}\left(1 - \frac{5}{2\pi}\arcsin\frac{1}{16}\sqrt{110-2\sqrt{145}}\right) \\
\varphi &= 2.87\cdot 10^4\,\mathrm{V\cdot m}\,,
\end{aligned}$$

where we utilized some trigonometric identities for arcsine.

Numerical solution

It is clear that the solution presented above is algebraically demanding and lengthy. Since we only need the numerical result, the problem can also be solved numerically. For example, in the program Geogebra, we can project a triangle constituting one tenth of the pentagram onto a sphere with its centre at the point charge, which passes through its tips (see the picture).

Let's label its radius r. Geogebra will calculate the lengths of circular arcs a, b, and c, from which we can, using the equation

$$\tan\left(\frac{E}{4}\right) = \sqrt{\tan\left(\frac{r}{2}\right)\tan\left(\frac{r-a}{2}\right)\tan\left(\frac{r-b}{2}\right)\tan\left(\frac{r-c}{2}\right)}\,, \tag{10}$$

calculate the solid angle E that is taken up by the triangle from the perspective of the point charge. On a sphere, the flux is constant, so it is sufficient to multiply the area by the electric field strength in the distance r.

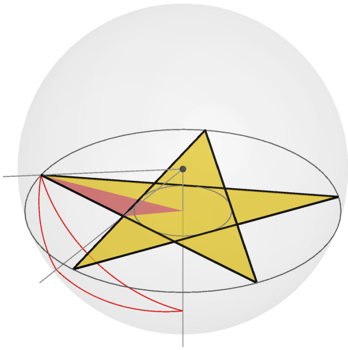

Fig. 3: Geometry of charge and pentagram

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Problem M.1 ... Rick's microplanet 3 points

Imagine that you are hiding from the Galactic Federation on a planet which has the same gravitational acceleration on the equator as the Earth ($a_g = 9.83\,\mathrm{m\cdot s^{-2}}$), as well as the same centrifugal acceleration on the equator ($a_c = 0.034\,\mathrm{m\cdot s^{-2}}$), but its radius is only $R = 5\,\mathrm{km}$. How long would one day on this planet be?

Consider the planet to be an ideal homogeneous sphere. The period of the planet's orbit around its star is negligible compared to the length of the day.

Karel and Matěj were wondering where Rick and Morty's family escaped.

Thanks to the fact that we can neglect the period of the planet's orbit, we can simply calculate the length of the day using the centrifugal force $F_c = ma_c = mv_r^2/R = m/omega_r^2 R$, where v_r is the velocity of a point on the planet's equator and ω_r is the angular velocity of its rotation. We can express the angular velocity from the period of rotation T as $\omega_r = 2\pi/T$. We get

$$a_c = \frac{4\pi^2}{T^2}R \qquad \Rightarrow \qquad T = 2\pi\sqrt{\frac{R}{a_c}} \doteq 40\,\mathrm{min}\,.$$

The period of the planet's rotation, i.e. the length of one day there, would be 40 minutes.

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Problem M.2 ... interdimensional density 3 points

Imagine that you are still hiding on the planet from the previous problem. This planet has the same gravitational acceleration on the equator ($a_\mathrm{g} = 9.83\,\mathrm{ms^{-2}}$) and the same centrifugal acceleration ($a_\mathrm{c} = 0.034\,\mathrm{ms^{-2}}$) as the Earth, but its radius is only $R = 5\,\mathrm{km}$. What is the density of this planet?

Assume that the planet is an ideal homogeneous sphere.

Karel was watching where Rick and Morty's family escaped.

We obtain the density of the planet from the gravitational acceleration on its surface

$$a_g = G\frac{M}{R^2}$$

where $G = 6.67\cdot 10^{-11}\,\mathrm{kg^{-1}\cdot m^3\cdot s^{-2}}$ is the gravitational constant. The mass can be expressed as $M = \varrho V \; = 4\pi R^3\varrho/3$, where V is the volume of the planet.

$$a_g = G\frac{4\pi}{3}\varrho R \qquad \Rightarrow \qquad \varrho = \frac{3}{4\pi}\frac{a_g}{GR} \doteq 7.04\cdot 10^6\,\mathrm{kg\cdot m^{-3}}$$

The density of the planet would have to be about 700 times higher than the density of water. By the way, this corresponds to the total mass of the planet

$$M = \frac{4}{3}\pi\varrho R^3 = \frac{4}{3}\pi R^3\frac{3}{4\pi}\frac{a_g}{GR} = \frac{a_g R^2}{G} \doteq 3.7\cdot 10^{18}\,\mathrm{kg}\,,$$

which would be more than 6 orders of magnitude smaller than the mass of our Earth.

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Problem M.3 ... squanchy frisbee 4 points

Consider the planet from the previous problems. This planet has the same gravitational acceleration on the equator ($a_\mathrm{g} = 9.83\,\mathrm{ms^{-2}}$) and the same centrifugal acceleration ($a_\mathrm{c} = 0.034\,\mathrm{ms^{-2}}$) as the Earth, but its radius is only $R = 5\,\mathrm{km}$. Life on the planet is boring, so you throw an object eastward along the equator. You want it to travel around the planet and return back to you from the west. How long does it take the object to come back?

Assume that the planet is an ideal homogeneous sphere and air drag is negligible. On this planet, the stars disappear on the western horizon. *Matej had nobody to play with.*

For the object to be able to orbit around the planet, it has to move with the first cosmic velocity, which is

$$v_\mathrm{k} = \frac{GM}{R} = \sqrt{a_\mathrm{g}R}\,,$$

where we expressed the mass of the planet as $M = \frac{a_\mathrm{g}R^2}{G}$. Don't forget that when we stand at the equator, we are rotating together with the planet, with the velocity $v_\mathrm{o} = \sqrt{a_o R}$. Try to think about why are rotating eastward. This means that we need to subtract our velocity from the absolute velocity of the object v_k. The time it takes the object to travel around the planet is

$$t = \frac{2\pi\sqrt{R}}{\sqrt{a_g} - \sqrt{a_o}} = 150.6\,\mathrm{s}\,.$$

By the way, the object would need to be thrown with the velocity $v = \sqrt{a_\mathrm{g}R} - \sqrt{a_\mathrm{o}R} = 208\,\mathrm{m\cdot s^{-1}}$, which is more than two times faster than the fastest recorded tennis serve.

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Problem M.4 ... wubba lubba dub dub — 4 points

We'll stay on the planet from the previous problems. This planet has the same gravitational acceleration on the equator ($a_\mathrm{g} = 9.83\,\mathrm{ms^{-2}}$) and the same centrifugal acceleration ($a_\mathrm{c} = 0.034\,\mathrm{ms^{-2}}$) as the Earth, but its radius is only $R = 5\,\mathrm{km}$. How high above the surface would the orbit of a geostationary satellite be, if it existed at all? If it does not exist, fill in 0 as the result.

Assume that the planet is an ideal homogeneous sphere.

Karel was watching where Rick and Morty's family escaped.

Let's focus on the geostationary orbit above the equator. A satellite on such an orbit is staying above one fixed place. We'll start from the equality of the gravitational and centrifugal force. Using this, we'll determine the distance from the center of the planet r for which a circular orbit would be geostationary. If it is smaller than R, then the planet does not have a geostationary orbit. If it is greater, we can determine the distance from the surface as $r - R$.

$$m\omega^2 r = G\frac{mM}{r^2} \qquad \Rightarrow \qquad \frac{4\pi^2}{T^2} = \frac{GM}{r^3} \qquad \Rightarrow \qquad r^3 = \frac{GMT^2}{4\pi^2}$$

We need to express r using the given quantities:

$$r^3 = \frac{G}{4\pi^2}\frac{a_g R^2}{G}\frac{4\pi^2 R}{a_o} = \frac{a_g}{a_c}R^3 \qquad \Rightarrow \qquad r = \sqrt[3]{\frac{a_g}{a_c}}R \doteq 33\,\mathrm{km}\,.$$

A geostationary orbit does exist, at the height approximately 28 km above the surface of our planet.

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Problem E.1 ... breakers — 3 points

The FYKOS-bird has two cylindrical copper wires, each with radius $r = 1\,\mathrm{mm}$ and resistivity $\varrho = 1.69\cdot 10^{-8}\,\Omega\cdot\mathrm{m}$. He wants to make an extension cord, but he connects two ends of one wire to a socket by mistake, so the wire short-circuits the socket. What is the maximum length of the wire which activates the circuit breaker? The voltage in the socket is 230 V and the maximum allowed current through the breaker is 16 A. *Mišo short-circuited.*

Let's use the formula for the resistance of a wire with cross-sectional area S and length l

$$R = \frac{\varrho l}{S} = \frac{\varrho l}{\pi r^2}\,,$$

where we used $S = \pi r^2$. The current can be expressed from Ohm's law

$$I = \frac{U}{R} = \frac{\pi r^2 U}{\varrho l} \qquad \Rightarrow \qquad l = \frac{\pi r^2 U}{\varrho I} \doteq 2\,670\,\mathrm{m}\,,$$

where we substituted for the voltage in the socket and the critical value of the current. If we stick the wire in the socket and its length is greater than 2 670 m, nothing should happen. You can try that at home if you don't believe it.

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Problem E.2 ... Fykonacci 3 points

The FYKOS-bird was playing with some resistors. He connected two points A and B by a resistor with resistance $R_1 = 1.600\,\Omega$. He thought it was too much, so he added another resistor with the same resistance $R_2 = 1.600\,\Omega$ between these points, parallel to the first one. Yet, he was still not satisfied, so he added a third resistor $R_3 = R_1 + R_2 = 3.200\,\Omega$, parallel to the others. Then he went bonkers and started adding more and more resistors parallel to the previous ones, with resistances $R_4, R_5, \dots$ described by the formula $R_n = R_{n-2} + R_{n-1}$, where $n \in N$. What was the final resistance between points A and B?

Jáchym does not enjoy physics, so he tries to create some math problems.

Let us denote the n-th element of the Fibonacci sequence by F_n, where $F_1 = 1$ and $F_2 = 1$. Therefore $R_n = F_n R_1$. All the resistors are connected in parallel, which means that the total resistance R satisfies

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \cdots = \sum_{n=1}^{\infty} \frac{1}{R_n} = \frac{1}{R_1} \sum_{n=1}^{\infty} \frac{1}{F_n}\,.$$

The Fibonacci sequence increases roughly as an exponential curve, so it is no surprise that the sum of its reciprocal values converges. We don't need to calculate it analytically, its value can be found on the internet,[9] or we can estimate it using numerical methods. In any case, the result is

$$\sum_{n=1}^{\infty} \frac{1}{F_n} \doteq 3.359886\dots\,.$$

Now we can easily see that the total resistance is $R = 0.4762\,\Omega$.

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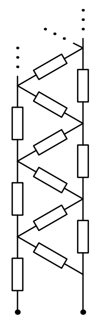
Problem E.3 ... triangular ladder 4 points

The FYKOS-bird wanted to climb onto a certain resistive skyscraper which you might also encounter. Therefore, he built a triangular ladder (infinite, of course), made wholly of resistors with resistance $R = 1.0\,\Omega$. Calculate the resistance between the legs of the ladder.

Lego was infinitely sad because none of his other tasks made it through.



The principle of solving tasks with infinite resistive networks is to denote the result by R_v. Then we only find a part of the ladder which is the same as the original ladder, while having in mind that the ladder is infinite.

[9] https://en.wikipedia.org/wiki/Fibonacci_number

In our case, if we disconnect the first vertical resistor and the first diagonal resistor, the final diagram will be the same (just mirrored, but the resistance does not depend on this). Even if we cut away another 2 resistors, the result stays the same, we will just spend more time on cutting. This infinite network will have a resistance R_v. When we connect a resistor with resistance R_v back to the other two resistors (i.e. parallel to the diagonal one), we will get a circuit with the same resistance as the original one. We get the equation

$$R_\mathrm{v} = R + \frac{RR_\mathrm{v}}{R + R_\mathrm{v}}\,,$$
$$0 = R_\mathrm{v}^2 - RR_\mathrm{v} - R^2\,.$$

This is a quadratic equation, so we can solve it easily by using the discriminant formula. Among the two roots we get, we naturally care only about the positive one, whose value is

$$R_\mathrm{v} = \frac{1+\sqrt{5}}{2} R \doteq 1.62\,R\,,$$

where the value of the fraction is called the golden ratio.

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Problem E.4 ... nonlinear FYKOS-bird 4 points

The FYKOS-bird was tired of connecting resistors and so he decided to connect himself in series with two batteries, each with voltage $U_\mathrm{b} = 4.50\,\mathrm{V}$, and a resistor with resistance $R = 50.0\,\Omega$. The bird behaves as a non-linear component with the current–voltage characteristic shown in the figure 4. The cut-in voltage of the bird is $U_0 = 1.50\,\mathrm{V}$, the saturation voltage of the bird is $U_\mathrm{n} = 6.00\,\mathrm{V}$ and the saturation current through the bird is $I_\mathrm{n} = 250\,\mathrm{mA}$. In each of the three parts of the current–voltage characteristic, the bird behaves as a linear component. Determine the current flowing through the bird. *Dodo remembered labs at midnight.*

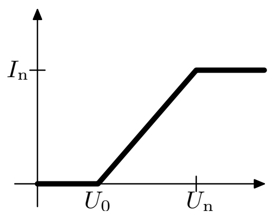
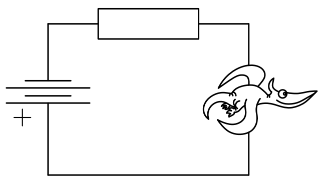


Fig. 4: I-V curve



The voltage across the circuit $U_\mathrm{z} = 2U_\mathrm{b} = 9.0\,\mathrm{V}$ is divided between the voltage across the resistor U_R and the voltage across the FYKOS-bird U_F.

$$U_\mathrm{z} = U_R + U_\mathrm{F}\,.$$

Ohm's law applies to the voltage across the resistor, and hence $U_R = RI$, where I is the current through the circuit. Expressing I in terms of other variables, we have

$$I(U_\mathrm{F}) = \frac{U_\mathrm{z} - U_\mathrm{F}}{R}\,,$$

where $I(U_F) = I$ is the current through the bird given by its current–voltage characteristic. The simplest solution to the problem is a graphical one. Then we only need to find the intersection of the current–voltage curve with the line $I(U) = \frac{U_\mathrm{z}-U}{R}$. From the graphical solution, it is clear that this intersection lies within the second part of the characteristic, which is described by the equation

$$I(U_\mathrm{F}) = I_\mathrm{n}\frac{U_\mathrm{F} - U_0}{U_\mathrm{n} - U_0}\,.$$

After substitution, we arrive at the result

$$\frac{U_\mathrm{z} - U_\mathrm{F}}{R} = I_\mathrm{n}\frac{U_\mathrm{F} - U_0}{U_\mathrm{n} - U_0}\,,$$

$$U_\mathrm{F} = \frac{I_\mathrm{n} R U_0 + U_\mathrm{z}\left(U_\mathrm{n} - U_0\right)}{I_\mathrm{n} R + U_\mathrm{n} - U_0}\,,$$

from which we get the current

$$I = I_\mathrm{n}\frac{U_\mathrm{z} - U_0}{I_\mathrm{n} R + U_\mathrm{n} - U_0} \doteq 110\,\mathrm{mA}\,.$$

Problem X.1 ... Chernobyl 3 points

Soon after the reactor no. 4 exploded, a radiation of 5 roentgens per hour was measured at a distance of $800\,\mathrm{m}$ *from the reactor. How much would have been measured at the distance of* $200\,\mathrm{m}$ *from the reactor? Neglect the effects of air and environment on the propagation of radiation.* *Matěj was watching the series.*

Let us begin with the fact that any dose of radiation is inversely proportional to the square of distance. Therefore, when the distance from the source is multiplied by 4, the dose decreases 16 times, so

$$16 \cdot 5\,\mathrm{R{\cdot}h^{-1}} = 80\,\mathrm{R{\cdot}h^{-1}}\,.$$

The radiation at the distance of $200\,\mathrm{m}$ equals $80\,\mathrm{R{\cdot}h^{-1}}$.

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Problem X.2 ... hold my graphite 3 points

An unsuspecting firefighter grabs a piece of graphite with an activity of $1.00 \cdot 10^{10}\,\mathrm{Bq}$*, lying near the now defunct reactor. How many particles will go into (or through) his hand if the firefighter holds the graphite for* 10 seconds*? Consider the chunk of graphite to be a sphere with a diameter of* 5 cm *and assume that the firefighter's hand covers one fifth of its surface when holding it. Assume further that each particle which leaves the piece of graphite corresponds to the decay of one nucleus.* *Matěj was chilling while watching memes.*

The unit becquerel tells us how many radiation particles a given object emits per one second. We can simply multiply the activity by time, and of course, we cannot forget the factor of one fifth, which takes into consideration the part of the surface covered by the hand

$$\frac{1}{5} \cdot 10^{10}\,\mathrm{Bq} \cdot 10\,\mathrm{s} = 2 \cdot 10^{10}\,.$$

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Problem X.3 ... not great, not terrible 3 points

Calculate the activity of a destroyed nuclear reactor. We measure 10000 radiation particles per minute per square decimeter at a distance of 5.00 km *from the reactor. Neglect the effects of air and environment on the propagation of radiation.* *Matěj wanted to set the answer to* 3.6 R.

The activity of an emitter indicates the number of emitted radiation particles per a given time unit, so we get $A = (10\,000\,\mathrm{min^{-1}{\cdot}dm^{-2}}) \cdot 4\pi(5\,\mathrm{km})^2 \doteq 5.24 \cdot 10^{12}\,\mathrm{s^{-1}}$.

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Problem X.4 ... radio-activity 3 points

By how many percent is the radiation around a destroyed nuclear power plant reduced after a 2.00*m thick sarcophagus is built around it? Assume that* 3.00 dm *of the material which the sarcophagus is made of can capture, on average,* 50.0 % *of the radiation.*
Matěj is worried about his safety.

Only

$$0.5^{2/0.3} \doteq 0.0098$$

of the original radiation passes through the sarcophagus, so radiation drops by 99.0 %. Let us add that the sarcophagus is primarily built to prevent radioactive material from leaking into the air.

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